

POINTS ON VARIETIES
EXERCISES FOR THE COURSE

- (1) Prove that

$$N_{\mathbb{P}^n}(B) := \#\{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leq B\} \sim \frac{2^n}{\zeta(n+1)} B^{n+1},$$

as $B \rightarrow \infty$, where H is the standard exponential height function on $\mathbb{P}^n(\mathbb{Q})$, equipped with the sup-norm. What error term do you get when $n = 1$?

- (2) Let $\nu_s(B)$ denote the number of $(a_1, \dots, a_s), (b_1, \dots, b_s) \in \mathbb{N}^s$ such that

$$a_1^3 + \dots + a_s^3 = b_1^3 + \dots + b_s^3$$

and $a_i, b_i \leq B$.

- (a) Show that $\nu_2(B) = O_\varepsilon(B^{2+\varepsilon})$, for any $\varepsilon > 0$.
 (b) It follows from Hua's lemma that $\nu_4(B) = O_\varepsilon(B^{5+\varepsilon})$, for any $\varepsilon > 0$. Deduce that $\nu_3(B) = O_\varepsilon(B^{3.5+\varepsilon})$.
 (c) What is the best lower bound you can prove for the number $N_X(B)$ of points of bounded height on the Fermat cubic threefold $X = \{x_1^3 + \dots + x_5^3 = 0\} \subset \mathbb{P}^4$?
 (d) Same question for the cubic surface $X = \{x_1^3 + x_2^3 + x_3^3 + 2x_4^3 = 0\} \subset \mathbb{P}^3$.
- (3) Let $e_q(\cdot) = \exp(\frac{2\pi i}{q}\cdot)$ and let $F \in \mathbb{Z}[x_1, \dots, x_n]$ is a homogeneous polynomial.

- (a) Let

$$S_q = \sum_{\mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^n} \sum_{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n} e_q(aF(\mathbf{x})).$$

Prove that S_q is a multiplicative function of q .

- (b) Prove that

$$S_{p^e} = p^e N(p^e) - p^{n-1+e} N(p^{e-1}),$$

for any prime power p^e , where $N(p^e) = \#\{\mathbf{x} \in (\mathbb{Z}/p^e\mathbb{Z})^n : F(\mathbf{x}) \equiv 0 \pmod{p^e}\}$.

- (c) Use Gauss sums to calculate S_p when $F(\mathbf{x}) = x_1^2 + \dots + x_n^2$ and $p > 2$.
- (4) (a) Use the large sieve to prove that $\#\{n \leq N : n \text{ is a square}\} = O(\sqrt{N})$.
 (b) What bound does the large sieve give for $\#\{n \leq N : n \text{ is square-full}\}$? What is the true asymptotic behaviour of this counting function, as $N \rightarrow \infty$?
- (5) Let $Y \subset \mathbb{A}_{\mathbb{Z}}^n$ be a closed subscheme of codimension at least 2 and let $X = \mathbb{A}_{\mathbb{Z}}^n \setminus Y$.
 (a) Working in the product topology, let $W = \prod_p W_p \subset \prod_p X(\mathbb{Z}_p)$ be a non-empty open subset with $W_p = X(\mathbb{Z}_p)$ for all but finitely many primes. Let S be a finite set of primes such that $W_p = X(\mathbb{Z}_p)$ for $p \notin S$. Show that for every sufficiently large integer M , there exists $Q \in \mathbb{N}$ and a coset $\Lambda = \{\mathbf{a} + Q\mathbb{Z}^n\}$ such that every $\mathbf{x} \in \Lambda$ satisfies $\mathbf{x} \in W_p$ for $p \in S$ and $\mathbf{x} \in X(\mathbb{Z}_p)$ for all $p \leq M$.
 (b) Use the Ekedahl sieve to deduce that there exists $\mathbf{x} \in \Lambda$ such that $\mathbf{x} \in X(\mathbb{Z}_p)$ for all $p > M$.
 (c) Deduce that $X(\mathbb{Z})$ is dense in $\prod_p X(\mathbb{Z}_p)$ with respect to the product topology).
 (d) Let $\mathbb{G}_m = \mathbb{A}_{\mathbb{Z}}^1 \setminus \{0\}$. Does the strong approximation statement hold here?
 (e) Look-up *arithmetic purity*.
- (6) Let $Q \subset \mathbb{A}_{\mathbb{Z}}^n$ be a quadric hypersurface. What Ekedahl sieve statement would you expect for points on this quadric? Suppose that Q is given by $x_1x_2 = x_3x_4$ and let $L \subset Q$ be the linear space $x_1 = x_2 = x_3 = 0$. Explain why this example is problematic.