POINTS ON VARIETIES

EXERCISES FOR THE COURSE

(1) Prove that
$$N_{\mathbb{P}^n}(B) := \#\{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leqslant B\} \sim \frac{2^n}{\zeta(n+1)} B^{n+1},$$
as $B \to \infty$, where H is the standard exponential height function on $\mathbb{P}^n(\mathbb{Q})$, equipped with the sup-norm. What error term do you get when $n = 1$?

(2) Let $\nu_s(B)$ denote the number of $(a_1, \dots, a_s), (b_1, \dots, b_s) \in \mathbb{N}^s$ such that
$$a_1^3 + \cdots + a_s^3 = b_1^3 + \cdots + b_s^3$$
and $a_i, b_i \leqslant B$.
 (a) Show that $\nu_2(B) = O_\varepsilon(B^{2+\varepsilon})$, for any $\varepsilon > 0$.
 (b) It follows from Hua's lemma that $\nu_4(B) = O_\varepsilon(B^{5+\varepsilon})$, for any $\varepsilon > 0$. Deduce that $\nu_3(B) = O_\varepsilon(B^{3.5+\varepsilon})$.
 (c) What is the best lower bound you can prove for the number $N_X(B)$ of points of bounded height on the Fermat cubic threefold $X = \{x_1^3 + \cdots + x_5^3 = 0\} \subset \mathbb{P}^4$?

(3) Let $e_q(\cdot) = \exp(\frac{2\pi i}{q} \cdot)$ and let $F \in \mathbb{Z}[x_1, \dots, x_n]$ is a homogeneous polynomial.
 (a) Let
$$S_q = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \sum_{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^n} e_q\left(aF(\mathbf{x})\right).$$
 Prove that S_q is a multiplicative function of q.
 (b) Prove that
$$S_{p^e} = p^e N(p^e) - p^{n-1+e} N(p^{e-1}),$$
 for any prime power p^e, where $N(p^e) = \#\{\mathbf{x} \in (\mathbb{Z}/p^e\mathbb{Z})^n : F(\mathbf{x}) \equiv 0 \bmod p^e\}$.
 (c) Use Gauss sums to calculate S_p when $F(\mathbf{x}) = x_1^2 + \cdots + x_n^2$.

(4) (a) Use the large sieve to prove that $\#\{n \leqslant N : n \text{ is a square}\} = O(\sqrt{N})$.
 (b) Use the large sieve to prove that $\#\{n \leqslant N : n \text{ is square-full}\} = o(N)$. What is the true asymptotic behaviour of this counting function?

(5) Let $Y \subset \mathbb{A}^n_{\mathbb{Z}}$ be a closed subscheme of codimension at least 2 and let $X = \mathbb{A}^n_{\mathbb{Z}} \setminus Y$.
 (a) Working in the product topology, let $W = \prod_p W_p \subset \prod_p X(\mathbb{Z}_p)$ be a non-empty open subset with $W_p = X(\mathbb{Z}_p)$ for all but finitely many primes. Let S be a finite set of primes such that $W_p = X(\mathbb{Z}_p)$ for $p \notin S$. Show that there exists an integer M and a coset $\Lambda = \{\mathbf{a} + M\mathbb{Z}^n\}$ such that every $\mathbf{x} \in \Lambda$ satisfies $\mathbf{x} \in W_p$ for $p \in S$ and $\mathbf{x} \in X(\mathbb{Z}_p)$ for all $p \leqslant M$.
 (b) Use the Ekedahl sieve to deduce that there exists $\mathbf{x} \in \Lambda$ such that also $\mathbf{x} \in X(\mathbb{Z}_p)$ for all $p > M$.
 (c) Deduce that $X(\mathbb{Z})$ is dense in $\prod_p X(\mathbb{Z}_p)$ (in the product topology).
 (d) Look-up *arithmetic purity.*

(6) Let $Q \subset \mathbb{A}^n_{\mathbb{Z}}$ be a quadric hypersurface. What Ekedahl sieve statement would you expect for points on this quadric? Suppose that Q is given by
$$x_1 x_2 = x_3 x_4$$
and let $L \subset Q$ be the linear space $x_1 = x_2 = x_3 = 0$. Explain why this example is problematic.