Selberg’s Central Limit Theorem

Mollifiers, Moments, and the Zeta Function — Lecture 3 of 3

Selberg's central limit theorem

Theorem 1 (Selberg)

Let V be a fixed real number. Then for all large T,

$$\frac{1}{T}\mathrm{meas}\left\{T \le t \le 2T : \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \ge V\right\} \sim \frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-u^2/2}\,du.$$

- In words: $\log|\zeta(\frac{1}{2}+it)|$ is approximately Gaussian with mean 0 and variance $\frac{1}{2}\log\log T$.
- Selberg's original proof is long and requires a strong zero-density theorem close to the half-line.
- In this lecture we give a complete (short!) proof following the approach of Radziwiłł–Soundararajan (2015), which avoids zero-density estimates entirely.
- The method is inspired by an earlier approach due to Laurinchikas which went through fractional moments.

Motivation from fractional moments

Recall from Lecture 2 that we showed

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \ll (\log T)^{k^2}.$$

- Rewriting $|\zeta|^{2k} = \exp(2k\log|\zeta|)$, this says the moment generating function of $\log|\zeta(\frac{1}{2}+it)|$ satisfies

$$\frac{1}{T}\int_T^{2T} \exp(2k\log|\zeta(\tfrac{1}{2}+it)|)\,dt \asymp \exp\left(k^2\log\log T\right).$$

- For a Gaussian $\mathcal{N}(0, \frac{1}{2}\log\log T)$ the MGF is $\exp(k^2\cdot\frac{1}{2}\log\log T)$.
- The CLT roughly corresponds to fractional moments with $k \asymp 1/\sqrt{\log\log T}$.

The prime sum heuristic

- For σ slightly to the right of $\frac{1}{2}$ and primes up to X:
$$\log\zeta(\sigma+it)=\mathcal{P}(\sigma+it)+\sum_{\substack{\rho=\beta+i\gamma\\ |\gamma-t|\leq 1/\log X,\ \beta>\sigma}}\log\frac{1}{(\sigma+it-\rho)}+O(1).$$
- If there are no zeros ρ with $\beta>\sigma$ very close to $\sigma+it$, the zero contribution is bounded and
$$\log|\zeta(\sigma+it)|\approx\mathfrak{R}\,\mathcal{P}(\sigma+it).$$
- The sum $\mathfrak{R}\sum_{p\leq X}p^{-1/2-it}$ is a sum of many approximately independent terms $\cos(t\log p)/\sqrt{p}$ with total variance $\sim\frac{1}{2}\log\log X$.
- By a standard CLT argument this sum is normally distributed. So if only primes matter, we get a Gaussian.

The four-step strategy

We prove that the chain of approximations

$$\log|\zeta(\tfrac{1}{2}+it)| \;\approx\; \log|\zeta(\sigma_0+it)| \;\approx\; \Re\,\mathcal{P}(\sigma_0+it)$$

holds for most t, and that the right-hand side is Gaussian.

- **Step 1.** $\log|\zeta(\frac{1}{2}+it)| \approx \log|\zeta(\sigma_0+it)|$ for $\sigma_0 = \frac{1}{2} + W/\log T$ (move off the line).
- **Step 2.** $\Re\,\mathcal{P}(\sigma_0+it)$ is Gaussian (method of moments).
- **Step 3.** There exists a short Dirichlet polynomial $M(s)$ such that $M \approx \exp(-\mathcal{P})$ and $\zeta M \approx 1$ for most t.
- **Step 4.** Chain: $\log|\zeta| \approx -\log M \approx \Re\,\mathcal{P}$, which is Gaussian.

Parameters

- Throughout we fix

$$W = (\log\log\log T)^4, \quad \sigma_0 = \tfrac{1}{2} + W/\log T.$$

- Two scales of primes:

$$Y = T^{1/(\log\log T)^2}, \qquad X = T^{1/(\log\log\log T)^2}.$$

- The prime sums (using von Mangoldt weights for convenience):

$$\mathcal{P}_1(s) = \sum_{2\le n\le Y} \frac{\Lambda(n)}{n^s \log n}, \qquad \mathcal{P}_2(s) = \sum_{Y<n\le X} \frac{\Lambda(n)}{n^s \log n}, \qquad \mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2.$$

- *Remark.* In Lecture 2 we wrote $\mathcal{P}_j = \Re \sum_p p^{-s}$; the two normalizations differ by negligible terms from prime powers p^k with $k \ge 2$.

Proposition 1 (Littlewood-type bound)

Proposition 1

Let T be large and $T \leq t \leq 2T$. For any $\sigma > 1/2$,

$$\int_{t-1}^{t+1} \big| \log|\zeta(\tfrac{1}{2}+iy)| - \log|\zeta(\sigma+iy)| \big| \, dy \ll (\sigma - \tfrac{1}{2}) \log T.$$

- This is the only place where zeros of ζ appear in the proof.

Proof sketch of Proposition 1

- Using the Hadamard factorization $\xi(s) = e^{A+Bs}\prod_\rho (1 - s/\rho)e^{s/\rho}$, we write

$$\log\left|\frac{\xi(\frac{1}{2}+iy)}{\xi(\sigma+iy)}\right| = \sum_\rho \log\left|\frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho}\right|.$$

- Integrating over $y \in (t-1, t+1)$: each zero $\rho = \beta + i\gamma$ contributes

$$\int_{t-1}^{t+1} \left|\log\left|\frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho}\right|\right| dy \ll \frac{\sigma - \frac{1}{2}}{1+(t-\gamma)^2}.$$

- Summing over zeros (using $\ll \log(|t|+k)$ zeros with $k \le |t-\gamma| < k+1$) gives the result.

Consequence for the CLT

- Proposition 1 gives a bound on average over a short interval. By Markov's inequality, for most $t \in [T, 2T]$ the pointwise bound holds:

$$\big|\log|\zeta(\tfrac{1}{2}+it)| - \log|\zeta(\sigma_0+it)|\big| \ll (\sigma_0 - \tfrac{1}{2})\log T = W.$$

- Since $W = (\log\log\log T)^4 = o(\sqrt{\log\log T})$, this error is negligible compared to the standard deviation $\sqrt{\frac{1}{2}\log\log T}$.
- So for most t: $\log|\zeta(\frac{1}{2}+it)| \approx \log|\zeta(\sigma_0+it)|$.

Takeaway

By Proposition 1, it suffices to prove the CLT for $\log|\zeta(\sigma_0+it)|$ instead of on the critical line itself.

Proposition 2 (Gaussian distribution of $\mathcal{P}$)

Proposition 2

As t varies in $[T, 2T]$, the distribution of $\Re(\mathcal{P}(\sigma_0 + it))$ is approximately normal with mean 0 and variance $\sim \frac{1}{2} \log\log T$.

- Proved by computing all moments of $\mathcal{P}_0(\sigma_0 + it) := \Re \sum_{p \le X} p^{-\sigma_0 - it}$ and matching them with the Gaussian.

Proof of Proposition 2 via moments

- **Key Lemma.** If $X^{k+\ell} \leq T$ then:
  - If $k \neq \ell$: $\int_T^{2T} \mathcal{P}_0^k \overline{\mathcal{P}_0}^\ell \, dt \ll T$.
  - If $k = \ell$: $\int_T^{2T} |\mathcal{P}_0|^{2k} \, dt = k! \, T(\log\log T)^k + O_k(T(\log\log T)^{k-1+\varepsilon})$.
- The proof expands $\mathcal{P}_0(s)^k = \sum_n a_k(n) n^{-s}$ and uses the mean value theorem for Dirichlet polynomials: diagonal terms $m = n$ dominate.
- For even k-th moments of $\Re \mathcal{P}_0$:

$$\frac{1}{T}\int_T^{2T} (\Re \mathcal{P}_0)^k \, dt = 2^{-k}\binom{k}{k/2}(k/2)! \, (\log\log T)^{k/2} + \text{lower order.}$$

- These are exactly the moments of $\mathcal{N}(0, \frac{1}{2}\log\log T)$. Since the Gaussian is determined by its moments, Proposition 2 follows.

What we need from the mollifier

- We need a short Dirichlet polynomial $M(s)$ with two properties:

Proposition 3

For most $t \in [T, 2T]$:

$$M(\sigma_0 + it) = (1 + o(1)) \exp\big(-\mathcal{P}_1(\sigma_0 + it) - \mathcal{P}_2(\sigma_0 + it)\big).$$

Proposition 4

$$\frac{1}{T} \int_T^{2T} |1 - \zeta(\sigma_0 + it) M(\sigma_0 + it)|^2 \, dt = o(1).$$

- Together these say: $\zeta \approx M^{-1} \approx \exp(\mathcal{P})$, so $\log|\zeta| \approx \Re\, \mathcal{P}$.

Proving Proposition 3: two-scale Taylor expansion

- The idea is exactly the multi-scale Taylor expansion from Lecture 2. On the typical set $|\mathcal{P}_j| \leq L_j$, we expand $\exp(-\mathcal{P}_j)$ in a Taylor polynomial of degree ℓ_j, obtaining a short Dirichlet polynomial $\mathcal{M}_j$.
- Scale 1 (primes $\leq Y$): variance $\sim \frac{1}{2} \log\log T$, Taylor degree $\ell_1 \sim (\log\log T)^2$, polynomial length $Y^{\ell_1} \leq T^{1/2}$.
- Scale 2 (primes in $(Y, X]$): variance $\sim \frac{1}{2} \log\log\log T$, Taylor degree $\ell_2 \sim (\log\log\log T)^2$, polynomial length $X^{\ell_2} \leq T^{1/2}$.
- The mollifier $M(s) = \sum_n \mu(n)a(n)/n^s$ is built from the same recipe: $a(n)$ restricts to squarefree n with $\leq \ell_1$ prime factors below Y and $\leq \ell_2$ between Y and X. Then $M = M_1 \cdot M_2$ where $M_j \approx \mathcal{M}_j \approx \exp(-\mathcal{P}_j)$ on the typical set.

Proving Proposition 4: $\zeta M \approx 1$

- This is the heart of the argument — it replaces the zero-density estimates used in Selberg's original proof.
- Since $\zeta(\sigma_0 + it) = \sum_{n \le T} n^{-\sigma_0 - it} + O(T^{-1/2})$ and M has leading coefficient 1:

$$\int_T^{2T} \zeta(\sigma_0 + it) M(\sigma_0 + it)\, dt = T + O(T^{1/2+\varepsilon}).$$

- Expanding $|1 - \zeta M|^2$, it suffices to show $\int_T^{2T} |\zeta M|^2 \, dt \sim T$.

Proving Proposition 4: the computation

- This is a mean-value computation: expand $|M|^2$, apply a classical formula for $\int (h/k)^{it}|\zeta(\sigma+it)|^2\,dt$ (Lemma 4 of Selberg), and evaluate the resulting Euler products.
- The key steps: the sum over mollifier coefficients factors over the two scales of primes, the truncation constraints can be removed with negligible error ($\ll (\log T)^{-90}$), and the main term assembles into

$$T\zeta(2\sigma_0)\prod_{p\le X}\left(1-\frac{1}{p^{2\sigma_0}}\right)=T\prod_{p>X}\left(1-\frac{1}{p^{2\sigma_0}}\right)^{-1}\sim T,$$

  since $(\sigma_0-\frac{1}{2})\log X = W/(\log\log\log T)^2=(\log\log\log T)^2\to\infty$, which makes $\sum_{p>X}p^{-2\sigma_0}$ negligible. The remaining terms are $o(T)$.

Proof of Selberg's CLT

The four propositions chain together:

1. **Proposition 4:** $\zeta(\sigma_0 + it)M(\sigma_0 + it) = 1 + o(1)$ for most t.
2. **Proposition 3:** $M(\sigma_0 + it) = (1 + o(1)) \exp(-\mathcal{P}(\sigma_0 + it))$ for most t.
3. **Combining:** $\zeta(\sigma_0 + it) \approx \exp(\mathcal{P}(\sigma_0 + it))$, so $\log|\zeta(\sigma_0 + it)| \approx \Re\, \mathcal{P}(\sigma_0 + it)$.
4. **Proposition 2:** $\Re\, \mathcal{P}(\sigma_0 + it)$ is Gaussian with mean 0, variance $\sim \frac{1}{2} \log\log T$.
5. **Proposition 1:** $\log|\zeta(\frac{1}{2} + it)| \approx \log|\zeta(\sigma_0 + it)|$ for most t. □

Advantages of this approach

- **No zero-density estimates.** The mean-value result (Proposition 4) is a complete substitute for the zero-density estimates used in Selberg's original proof and in the approach of Bombieri–Hejhal.
- **Transparent structure.** The proof cleanly separates: CLT for prime sums, mollifier $\approx \exp(-\mathcal{P})$, mollified mean value ≈ 1.
- **Connection to Lectures 1–2.** The mollifier M is a direct descendant of Bohr's idea of multiplying ζ by a partial Euler product (Lecture 1), and the two-scale Taylor expansion is the same device that gives $\int |\zeta|^{2k} \ll T(\log T)^{k^2}$ (Lecture 2).

What this method does not give

- **Imaginary part.** Proposition 4 gives $|\zeta| \approx |M|^{-1}$, but phases could differ by a multiple of 2π, so the CLT for $\arg\zeta(\frac{1}{2}+it)$ requires additional ideas.
- **Moment asymptotics.** This proof gives the CLT but not individual moment asymptotics that Selberg also established.

Extensions

- **Joint distribution.** The argument shows that $\log|\zeta(\frac{1}{2}+it)|$ and $\log|\zeta(\frac{1}{2}+it+i\alpha)|$ for fixed $\alpha \neq 0$ are asymptotically independent Gaussians.
- **Families of L-functions.** Propositions 1–3 generalize to higher-degree L-functions. Proposition 4 currently extends to degree ≤ 2 (via shifted convolution bounds), but is open for degree ≥ 3.
- **Quadratic twists.** For $L(\frac{1}{2},\chi_d)$ one can carry out Steps 2–4, but the analog of Proposition 1 is missing — we cannot exclude $L(\frac{1}{2},\chi_d)=0$ for a positive proportion of d.

The range of V and open questions

- Selberg's theorem holds for fixed V. It is conjectured to remain valid for $V = o(\sqrt{\log\log T})$, deep into the Gaussian tails.
- The approach here can be quantified to give rates of convergence to the Gaussian distribution.

What we covered

1. **Lecture 1:** Littlewood's formula, AM-GM, Bohr's idea, the mollifier $M(s) = \sum \mu(n)/n^s$, optimal coefficients and the Selberg sieve.
2. **Lecture 2:** Fractional moments $\int |\zeta|^{2k} \asymp T(\log T)^{k^2}$ via exponential weights, the multi-scale level set argument.
3. **Lecture 3:** Selberg's CLT via the Radziwiłł–Soundararajan approach: move off the critical line, show $\mathcal{P}$ is Gaussian, construct a mollifier with $\zeta M \approx 1$, conclude.

The unifying theme

Takeaway

Throughout all three lectures, the central object is the **mollifier** — a short Dirichlet polynomial that approximates $1/\zeta$. Whether we are counting zeros (Lecture 1), bounding moments (Lecture 2), or proving distributional results (Lecture 3), the key step is always: replace the Euler product by a tractable Dirichlet polynomial, then compute mean values.