The Concept of a Mollifier

Mollifiers, Moments, and the Zeta Function — Lecture 1 of 3

Littlewood's formula

- One of the main questions about the Riemann zeta-function is the location of its zeros.
- Fundamentally $\zeta(s)$ is an analytic function and the simplest and most direct approach to this question goes through Jensen's formula.
- In the context of $\zeta(s)$ this formula is known as Littlewood's formula and gives

$$\sum_{\substack{T \le \gamma \le 2T \\ \rho = \beta + i\gamma \\ \beta > \sigma}} (\beta - \sigma) = \int_T^{2T} \log |\zeta(\sigma + it)| \, dt + O(\log T).$$

The Euler product and cancellation

- Since ζ has an Euler product we expect,

$$\log|\zeta(\sigma+it)| \approx \sum_{p\le t} \frac{1}{p^{\sigma+it}}$$

and then

$$\int_T^{2T} \log|\zeta(\sigma+it)|\,dt = o(T).$$

- Note that this is a feature of the Euler product!

Dirichlet series without an Euler product

- For Dirichlet series $L(s)$ without an Euler product we generally don't expect

$$\int_T^{2T} \log |L(\sigma + it)| \, dt$$

to cancel out, instead the expression is roughly of size $\sim cT$.

- In fact for general Dirichlet series without an Euler product we usually expect $\sim c_{\alpha,\beta} T$ zeros in strips

$$\tfrac{1}{2} < \alpha \le \sigma \le \beta \ , \ T \le t \le 2T.$$

Almost-periodicity

- Usually it is also expected that Dirichlet series have a form of almost-periodicity, if one zero exists then there are $\approx T$ zeros from "periodicity".
- **Wild meta-conjecture:** If the number of zeros is $o(T)$ it's actually zero.

An open problem

Open problem

Can one create a Dirichlet L-series with abscissa of absolute convergence at $\Re s = 1$, analytic continuation up to $\Re s > \frac{1}{2}$, and a strip $\frac{1}{2} < \alpha \le \sigma \le \beta < 1$ such that $L(s)$ has at least one zero in the strip, but there is a sequence of $T \to \infty$ such that the number of zeros in

$$\tfrac{1}{2} < \alpha \le \sigma < \beta \, , \; 0 \le t \le T$$

is $o(T)$? Can one make the number of zeros go to infinity?

- Potential example: $L(s) = \frac{1}{\zeta(s)}$ although the zero is at $s = 1$ and even then we need to assume the Riemann Hypothesis for this to be a valid example.

Counting zeros via the logarithmic integral

Returning to ζ we have in general,

$$\varepsilon N(\sigma, T) \leq \int_T^{2T} \log|\zeta(\sigma + \varepsilon + it)|\, dt$$

where

$$N(\sigma, T) := \#\{\beta + i\gamma : \beta > \sigma\ ,\ T \leq t \leq 2T\ ,\ \zeta(\beta + i\gamma) = 0\}.$$

- We have no way of actually estimating the logarithmic integral above.

The AM-GM inequality

- The best known approach goes through the AM-GM inequality

$$\int_T^{2T} \log|\zeta(\sigma+it)|\,dt \le T\log\Big(\frac{1}{T}\int_T^{2T}|\zeta(\sigma+it)|^2\,dt\Big).$$

- The problem: If you recall the random model we expect,

$$\frac{1}{T}\int_T^{2T}|\zeta(\sigma+it)|^2\,dt \approx \mathbb{E}\Big[\Big|\prod_{p\le T}\Big(1-\frac{X(p)}{p^\sigma}\Big)^{-1}\Big|^2\Big] = \zeta(2\sigma).$$

Limitations of the AM-GM approach

- Therefore with this approach we can show that in any strip $\frac{1}{2} + \varepsilon < \sigma$ we have,

$$N(\sigma, T) \ll_{\varepsilon} T$$

  but we cannot get anything better.

- This upper bound is also valid for general L-series, we are therefore not using anything special about the Riemann ζ-function.
- **Key issue:** The second moment $\int |\zeta|^2$ sees the full Euler product and gives a constant $\zeta(2\sigma)$ that is > 1 but $O(1)$. We need to make the integrand closer to 1 on average.

Bohr's clever idea

Bohr had a clever idea to improve this upper bound.

- Notice that for any $X \leq T$ we have,

$$\int_T^{2T} \log|\zeta(\sigma+it)|\, dt = \int_T^{2T} \log\Big|\zeta(\sigma+it)\prod_{p\leq X}\Big(1-\frac{1}{p^{\sigma+it}}\Big)\Big|\, dt$$
$$+ \int_T^{2T} \Re\Big(\sum_{p\leq X}\frac{1}{p^{\sigma+it}}\Big)\, dt + O(1).$$

- The integral of the Dirichlet polynomial is actually very small of size $O(1)$.

Applying AM-GM to Bohr's decomposition

- If we now repeat the same approach as before we find,

$$\int_T^{2T} \log\left|\zeta(\sigma+it)\prod_{p\le X}\left(1-\frac{1}{p^{\sigma+it}}\right)\right| dt$$
$$\le T\log\left(\frac{1}{T}\int_T^{2T}\left|\zeta(\sigma+it)\prod_{p\le X}\left(1-\frac{1}{p^{\sigma+it}}\right)\right|^2 dt\right).$$

The random model prediction

- Now, what does the random model say? We expect,

$$\frac{1}{T}\int_T^{2T} \Big|\zeta(\sigma+it)\prod_{p\le X}\Big(1-\frac{1}{p^{\sigma+it}}\Big)\Big|^2 dt \approx \mathbb{E}\Big[\Big|\prod_{p>X}\Big(1-\frac{X(p)}{p^{\sigma}}\Big)^{-1}\Big|^2\Big]$$

and this is equal to

$$\sum_{\substack{n\ge 1\\ p|n \implies p>X}} \frac{1}{n^{2\sigma}} = 1 + O\Big(\sum_{n>X}\frac{1}{n^{2\sigma}}\Big) = 1 + O(X^{1-2\sigma}).$$

Consequences for the density of zeros

- Plugging this into our argument we therefore expect,

$$\int_T^{2T} \log|\zeta(\sigma+it)|\,dt \ll T\log\Big(1+O(X^{1-2\sigma})\Big) \ll TX^{1-2\sigma}.$$

- What are possible choices of X? If we can take any X, say $X = T^A$ we expect the Riemann Hypothesis.
- This hints at an actual result which is true, we'll get back to this later. But realistically what is the largest X that we can take?

The density hypothesis

- Since we have,

$$\zeta(\sigma + it) = \sum_{n \leq T} \frac{1}{n^{\sigma+it}} + O(1)$$

  we see that taking X beyond T might be challenging, we'll be in effect fighting terms that are not plainly contributing to ζ.

- If we take $X = T$ we recover the so-called **density hypothesis**:

$$\boxed{N(\sigma, T) \ll T^{2-2\sigma}.}$$

Primes in short intervals

- What are the implications of the density hypothesis?
- Two consequences: The right asymptotic for primes in $[x, x + x^{1/2+\varepsilon}]$ for every $\varepsilon > 0$ and x large enough, and the right asymptotic for primes in $[x, x + x^{\varepsilon}]$ for "almost every x" (i.e typical x).
- The proof of these results goes through writing,

$$\sum_{x \leq p \leq x + x^{1/2+\varepsilon}} \log p = x^{1/2+\varepsilon} + \sum_{|\rho| \leq \sqrt{x}} \frac{(x + x^{1/2+\varepsilon})^{\rho} - x^{\rho}}{\rho}$$

and then using the zero-density theorem to analyze the contribution of the zeros with β away from 1. For β very close to 1, one uses the zero-free region and other types of zero-density theorems.

Expanding the Euler product

- Can we make the idea of Bohr work out rigorously? We need to compute

$$\int_T^{2T} \left| \zeta(\sigma + it) \prod_{p \le X} \left(1 - \frac{1}{p^{\sigma+it}}\right) \right|^2 dt.$$

- Expanding the Euler product as a Dirichlet series,

$$\prod_{p \le X} \left(1 - \frac{1}{p^{\sigma+it}}\right) = \sum_{\substack{n \text{ sq-free} \\ p|n \implies p \le X}} \frac{\mu(n)}{n^{\sigma+it}}.$$

- Since such n satisfy $n \le e^X$, this is a Dirichlet polynomial of length e^X.

A heuristic reduction

- We proceed heuristically to first check if we even have a chance to bound this integral. It is reasonable to expect that $|\zeta(\sigma + it)| \ll 1$ on average inside the integral. If so, we are reduced to the presumably easier task of computing the mean square of a long Dirichlet polynomial:

$$\int_T^{2T} \Big| \sum_{n \le e^X} \frac{a(n)}{n^{\sigma+it}} \Big|^2 dt.$$

- To evaluate this we introduce a smooth majorant $\Phi \ge 0$ with $\Phi(x) \ge 1$ for $x \in [1, 2]$ and $\widehat{\Phi}$ compactly supported in $[-1, 1]$, getting the upper bound

$$\le \int_{\mathbb{R}} \Big| \sum_{n \le e^X} \frac{a(n)}{n^{\sigma+it}} \Big|^2 \Phi\Big(\frac{t}{T}\Big)\, dt.$$

Smoothing and Fourier analysis

- Expanding the square and computing the Fourier transform we get the identity

$$T\sum_{n,m}\frac{a(n)\overline{a(m)}}{(nm)^\sigma}\Big(\frac{n}{m}\Big)^{iT}\widehat{\Phi}\Big(T\log\frac{n}{m}\Big).$$

- Writing $n = m + h$, the weight $\widehat{\Phi}(T\log(n/m))$ is negligible unless $T|h|/m \lesssim 1$, so only $|h| \le e^X/T$ contributes. This gives

$$T\sum_{|h|\le e^X/T}\sum_{m\le e^X}\frac{a(m)\overline{a(m+h)}}{(m(m+h))^\sigma}\Big(\frac{m+h}{m}\Big)^{iT}.$$

Simplifying the expression

- For $m \approx e^X$ we simplify $(m(m+h))^\sigma \approx e^{2\sigma X}$. For the oscillatory factor, we use

$$\left(\frac{m+h}{m}\right)^{iT} = \left(1+\frac{h}{m}\right)^{iT} \approx e\left(\frac{Th}{m}\right)$$

where $e(x) = e^{2\pi i x}$. This gives roughly

$$Te^{-2\sigma X} \sum_{|h| \le e^X/T} \sum_{m \sim e^X} a(m)\overline{a(m+h)}\, e\left(\frac{Th}{m}\right).$$

Bohr's result

- If $e^X/T \leq 1$ (i.e. $X \leq \log T$) then only $h = 0$ contributes and everything is computable. For larger X we would need the self-correlations $\sum a(m)\overline{a(m+h)}$ of very smooth integers, which are out of reach.
- With $X \leq \log T$ we obtain:

$$\boxed{N(\sigma, T) \ll T(\log T)^{1-2\sigma} = o(T).}$$

Takeaway

The first result showing ζ has $o(T)$ zeros in any strip $\sigma > \frac{1}{2} + \varepsilon$, due to Bohr. The limitation: the Euler product produces a polynomial of length e^X, forcing $X \leq \log T$.

The inefficiency of the Euler product

What limits us from going further?

- Well one of the problems is that the use of the Euler product is very inefficient as it produces a very long but very sparse Dirichlet polynomial.
- This is similar to what one encounters in the sieve, if one wants to sieve using,

$$\prod_{p \le X} \left(1 - \mathbf{1}_{p|n}\right)$$

  it leads to sieve weights supported on integers up to e^X forcing one to choose $X \le \log N$ and proving that

$$\pi(N) \ll \frac{N}{\log\log N} = o(N).$$

Introducing the mollifier $M(s)$

An idea that is similar to the sieve is to replace the Euler product by a truncated Dirichlet polynomial with the Möbius function.

- Let
$$M(s) := \sum_{n \leq X} \frac{\mu(n)}{n^s}.$$
- Unfortunately the earlier trick won't work now so we proceed differently.

Zeros of ζ versus zeros of ζM

- Given an L-series denoted L, denote by $N_L(\sigma, T)$ the number of zeros of L in the box

$$\beta > \sigma \ , \ T \leq \gamma \leq 2T \ , \ L(\beta + i\gamma) = 0.$$

- It is obvious that the number of zeros of ζ is at most the number of zeros of ζM (every zero of ζ is a zero of ζM, but M may introduce new ones). We therefore have the inequality,

$$\varepsilon N_\zeta(\sigma, T) \leq \varepsilon N_{\zeta M}(\sigma, T) \leq \int_T^{2T} \log |\zeta(\sigma + it) M(\sigma + it)| \, dt.$$

The mollified mean-value problem

- We can now apply the AM-GM trick and bound the above by

$$T \log \Big(\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it) M(\sigma + it)|^2 \, dt \Big).$$

- It then remains to compute,

$$\int_T^{2T} |\zeta(\sigma + it) M(\sigma + it)|^2 \, dt.$$

- The key advantage: the mollifier $M(s)$ is a Dirichlet polynomial of length X, not e^X. This is what will let us break through the $\log T$ barrier.

Generally we call $M(s)$ a mollifier

Generally we call $M(s)$ a mollifier and it will be a key player in these lectures.

- There is a deep connection between mollifiers and sieves.
- For example an important question is to minimize

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)M(\tfrac{1}{2}+it)|^2\,dt.$$

- Notice that the coefficients of $\zeta(s)M(s)$ are

$$\sum_n \frac{1}{n^s}\Big(\sum_{d|n} a(d)\Big).$$

The optimal coefficients

- By Plancherel this is equivalent to minimizing

$$\sum_{x \leq n \leq x+h} \Big(\sum_{d \mid n} a(d) \Big) - h \sum_{d} \frac{a(d)}{d}$$

for almost all intervals x. Moreover by working out orthogonality this amounts to minimizing,

$$\sum_{n \leq X} \Big(\sum_{d \mid n} a(d) \Big)^2 .$$

- The coefficients that minimize this are,

$$\boxed{a(d) = \mu(d) \cdot \Big(1 - \frac{\log d}{\log X} \Big).}$$

Connection to the Selberg sieve

- In fact this problem led Selberg to invent the Selberg sieve, which is exactly the upper bound
$$\mathbf{1}_{n \text{ prime}} \leq \Big(\sum_{d|n} a(d)\Big)^2$$
with the same coefficients as above.

Takeaway

The optimal mollifier coefficients and the optimal sieve weights are the same object. Both problems ask for a short Dirichlet polynomial that approximates $1/\zeta$ as well as possible.

What we covered

1. **Littlewood's formula** reduces zero-counting to bounding $\int \log|\zeta|$.
2. **AM-GM** converts the problem to bounding a mean value, but gives only $N(\sigma, T) \ll T$.
3. **Bohr's idea:** multiply by the partial Euler product to kill the large primes. Limited to $X \le \log T$ giving $N(\sigma, T) \ll T(\log T)^{1-2\sigma}$.
4. **The mollifier** $M(s) = \sum_{n \le X} \mu(n)/n^s$ replaces the Euler product with a short polynomial, breaking through the $\log T$ barrier.
5. **Optimal coefficients** on the half-line connect to the Selberg sieve.

Next lectures

- **Lecture 2:** Salvage the idea of Bohr, construct a mollifier that works as an Euler product "most of the time" and then an application of these ideas to fractional moments.
- **Lecture 3:** Once we have the fractional moment worked out we point out that taking $k \to 0$ is hitting the regime of the central limit theorem and in fact that we expect a central limit theorem. Then prove it using these ideas.