Mollifiers, Moments, and the Zeta Function — Exercise Sheet

Lecture 1

Exercise 1 (Littlewood's lemma)

Prove that for $\sigma > \frac{1}{2}$ and T large,

$$\sum_{\substack{\rho=\beta+i\gamma \\ \beta>\sigma,\ T\le\gamma\le 2T}} (\beta-\sigma) = \frac{1}{2\pi}\int_T^{2T} \log|\zeta(\sigma+it)|\,dt + O(\log T).$$

Exercise 2 (The AM-GM step)

In the lecture we bounded $\int \log|\zeta M|$ using Jensen's inequality:

$$\int_T^{2T} \log|f|\,dt = \tfrac{1}{2}\int_T^{2T} \log|f|^2\,dt \le \tfrac{T}{2}\log\Big(\tfrac{1}{T}\int_T^{2T} |f|^2\,dt\Big).$$

What are the possible sources of loss?

Lecture 2

Exercise 3 (The typical level set)

Let $X(p)$ be independent and uniform on $|z|=1$, and set $V=\sum_{p\le X} 1/p$. Show that the contribution to $\mathbb{E}[\exp(\beta\,\Re\sum X(p)/\sqrt{p})]$ from values with $|\Re\sum X(p)/\sqrt{p}| > (\beta/2+\varepsilon)V$ is negligible.

Exercise 4 (Lower bounds for fractional moments)

The upper bound argument uses Hölder to write $\int|\zeta|^{2k} = \int|\zeta|^{2k}e^{-\gamma\mathcal{P}}\cdot e^{\gamma\mathcal{P}}$ and bounds each factor separately. Can you rearrange the argument to get a lower bound for $\int|\zeta(\frac{1}{2}+it)|^{2k}\,dt$?

Lecture 3

Exercise 5 (The twisted second moment)

For coprime $h, k \geq 1$ and $\sigma > 1$, guess an asymptotic formula for

$$\int_T^{2T} \Big(\frac{h}{k}\Big)^{it} |\zeta(\sigma + it)|^2 \, dt.$$

Exercise 6 (Moments of the prime sum)

Let $\mathcal{P}_0(s) = \sum_{p \leq X} p^{-s}$ with $X = T^{1/(\log\log\log T)^2}$. Show that

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_0(\tfrac{1}{2} + it)|^{2m} \, dt = m!\Big(\sum_{p \leq X} \frac{1}{p}\Big)^m + O_m((\log\log T)^{m-1+\varepsilon})$$

using the mean value theorem for Dirichlet polynomials. Deduce that the even moments of $\Re\mathcal{P}_0$ match those of $\mathcal{N}(0, \frac{1}{2}\log\log T)$, and explain why the method of moments gives Gaussianity.