EXERCISES FOR RANDOM GROUPS IN ARITHMETIC STATISTICS

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1. Lecture 1

(1) (1/Aut)
  (a) Let n be a positive integer and $[K : \mathbb{Q}] = n$. How many subfields of $\bar{\mathbb{Q}}$ are isomorphic to K, as a function of n and the automorphism group of K?
  (b) Let G_n be an Erdös-Rényi random graph, i.e. on n given verticies, each edge appears independently with probability $1/2$. What is the probability that G_n is isomorphic to a fixed (unlabelled) graph G_0?

(2) (Haar measure on $\hat{\mathbb{Z}}$ and $\mathbb{Z}_p$)
  (a) For the topology on $\hat{\mathbb{Z}}$ or $\mathbb{Z}_p$ generated by the (open) sets of elements congruent to a mod n for each a and n (where $n = p^k$ in the $\mathbb{Z}_p$ case), and the Borel σ-algebra for that topology, prove there is a unique measure ν, such that for all a and n, the set of elements congruent to a mod n has measure $1/n$. (Use Carathéodory's extension theorem or a similar result.)
  (b) Prove the measure above is a Haar measure, i.e. for any b in $\hat{\mathbb{Z}}$ or $\mathbb{Z}_p$, respectively, and any measurable set U, we have $\mu(U) = \mu(b + U)$.

(3) Find the sum of $|A|^{-1}|\operatorname{Aut} A|^{-1}|$ over all finite cyclic groups A, and conclude the probability that a group from the distribution where $\operatorname{Prob}(X \simeq A) = c_1|A|^{-1}|\operatorname{Aut} A|^{-1}$ is cyclic.

(4) (Random groups as cokernels of Haar matrices) Let n and u be positive integers and let $M_{n,n+u} \in \operatorname{Mat}_{n,n+u}(\hat{\mathbb{Z}})$ be a random matrix from Haar measure (i.e. each entry is independent from Haar measure on $\hat{\mathbb{Z}}$).
  (a) Fill in the details of Monday's proof, and generalize to any case of $u \geq 1$, to show that $\lim_{n\to\infty} \operatorname{Prob}(\operatorname{cok} M_{n,n+u} \simeq A) = c_u|A|^{-1}|\operatorname{Aut} A|^{-u}$, including finding the constant c_u.
  (b) For a fixed finite group B, find $\lim_{n\to\infty} \mathbb{E}(\#\operatorname{Sur}(\operatorname{cok} M_{n,n+u}, B))$.
  (c) Prove that a random group X with distribution $\operatorname{Prob}(X \simeq A) = c_u|A|^{-1}|\operatorname{Aut} A|^{-u}$ has moments $\mathbb{E}(\#\operatorname{Sur}(X, B)) = |B|^{-u}$. You can use the above arguments, but will need to interchange a sum and a limit, which will require application of a convergence theorem.
  (d) Let $M_{n,n+u,Z} \in \operatorname{Mat}_{n,n+u}(\mathbb{Z})$ be a random matrix with independent entries that are uniform random integers in $[-Z, Z]$. Find $\lim_{n\to\infty} \lim_{Z\to\infty} \mathbb{E}\#\operatorname{Sur}(\operatorname{cok} M_{n,n+u,Z}, B))$.

(5) ($u = 0$)
  (a) What goes wrong with the above arguments if $u = 0$?
  (b) Prove the analog of the previous problem with $u = 0$ but $\hat{\mathbb{Z}}$ replaced by $\mathbb{Z}_p$ (so $M_{n,n} : \mathbb{Z}_p^n \to \mathbb{Z}_p^n$).

(6) ($u < 0$) Let u be a negative integer. Let $M_{n,n+u} \in \mathrm{Mat}_{n,n+u}(\mathbb{Z}_p)$ be a random matrix from Haar measure. What kind of groups can appear as $\mathrm{cok}(M_{n,n+u})$? For each such possible group A, find $\lim_{n\to\infty} \mathrm{Prob}(\mathrm{cok}\, M_{n,n+u} \simeq A)$. (Hint: prove a non-probabilistic result that lets you determine whether $\mathrm{cok}\, M_{n,n+u} \simeq A$ from $(\mathrm{cok}\, M_{n,n+u})/p^{e+1}$, where p^e is the exponent of the torsion part of A.)

(7) ($\mathbb{F}_p$-vector spaces) Given X such that $\mathrm{Prob}(X \simeq A) = c_u |A|^{-1} |\mathrm{Aut}\, A|^{-u}$, for a fixed r, what is $\mathrm{Prob}(\dim_{\mathbb{F}_p} X/p = r)$?

(8) (Escape of mass) Give examples of random finite abelian groups $X_1, X_2, \dots$ such that

$$\sum_A \lim_{n\to\infty} \mathrm{Prob}(X_N \simeq A) < 1.$$

(9) (Moments not determining through limits) Give examples of random finite abelian groups $Y, X_1, X_2, \dots$ such that for every B,

$$\mathbb{E}(\mathrm{Sur}(Y, B)) = \lim_{n\to\infty} \mathbb{E}(\mathrm{Sur}(X_n, B)),$$

but at least for some A,

$$\mathrm{Prob}(Y \simeq A) \neq \lim_{n\to\infty} \mathrm{Prob}(X_n \simeq A),$$

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