# EXERCISE SHEET FOR MONTRÉAL SUMMER SCHOOL COURSE "INTRODUCTION TO RANDOM MULTIPLICATIVE FUNCTIONS", MAY 2026

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These exercises will be updated as the course goes on, this is *version 2*. Some of the questions will (hopefully!) only take a few minutes, others may require more thought.

## 1. Orthogonality

(1) Prove that if $f$ is a Rademacher random multiplicative function, and $n, m \in \mathbb{N}$, then

$$\mathbb{E}f(n)f(m) = \begin{cases} \mathbb{E}f(n)^2 = \mathbf{1}_{n \text{ is squarefree}} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

(Here $\mathbf{1}$ denotes the indicator function.)

(2) Determine the size of the second moment $\mathbb{E}|\sum_{n \le x} f(n)|^2$, where $f$ is a Rademacher random multiplicative function.

(3) (A bit harder) Find an expression for $\mathbb{E}f^*(n)f^*(m)$, where $f^*$ is an extended Rademacher random multiplicative function.

(4) If $f^*$ is an extended Rademacher random multiplicative function, then it turns out (very roughly) that

$$\sum_{n \le x} f^*(n) \approx \sqrt{x} \sum_{n \le x} \frac{f(n)}{\sqrt{n}},$$

where $f$ is a Rademacher random multiplicative function. Prove a precise statement of this kind. What does this suggest about any similarities/differences in the behaviour of $f^*$ and of $f$?

(5) (Harder) Using total multiplicativity and orthogonality, give the best upper bound you can for $\mathbb{E}|\sum_{n \le x} f(n)|^{2q}$, where $q \in \mathbb{N}$ and $f$ is a Steinhaus random multiplicative function. You should be able to get a bound of the form $x^q \log^{C(q)} x$, and will probably need to use some information about divisor functions.

If you are interested, see `https://arxiv.org/abs/1804.04114` for the real truth in this problem.

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## 2. Conditioning

(1) Let $P$ be a parameter, and let $\tilde{\mathbb{E}}$ denote expectation conditional on all the values $(f(p))_{p \leq P}$. Prove that if $f$ is a Steinhaus random multiplicative function, and $a_n$ are any complex numbers, then

$$\tilde{\mathbb{E}} \sum_{n \leq x} a_n f(n) = \sum_{\substack{n \leq x, \\ P(n) \leq P}} a_n f(n), \quad \text{and} \quad \tilde{\mathbb{E}} \left| \sum_{n \leq x} a_n f(n) \right|^2 = \sum_{\substack{m \leq x, \\ p | m \Rightarrow p > P}} \left| \sum_{\substack{n \leq x/m, \\ P(n) \leq P}} a_{nm} f(n) \right|^2 .$$

(2) Explicitly verify the Tower Property of conditional expectations: average the above expressions over all possible values of $(f(p))_{p \leq P}$ to confirm that

$$\mathbb{E} \sum_{n \leq x} a_n f(n) = a_1, \quad \text{and} \quad \mathbb{E} \left| \sum_{n \leq x} a_n f(n) \right|^2 = \sum_{n \leq x} |a_n|^2 .$$

(3) Suppose $p$ is a prime, and $\tilde{\mathbb{E}}$ denotes expectation conditional on all the values $(f(q))_{q < p}$, where $q$ runs over primes (note the strict inequality). Prove that

$$\tilde{\mathbb{E}} \sum_{\substack{n \leq x, \\ P(n) = p}} a_n f(n) = 0.$$

This observation allows one to write $\sum_{n \leq x} a_n f(n)$ as a *martingale* (filtered according to the largest prime factor), which is important to lots of distributional theory for sums of random multiplicative functions.

## 3. Tidying up

(1) Using Hölder's inequality, show that if we can prove the estimate

$$\mathbb{E} | \sum_{n \leq x} f(n) |^{2q} \asymp \left( \frac{x}{1 + (1-q)\sqrt{\log\log x}} \right)^q$$

for all $2/3 \leq q \leq 1$ (say), this implies the estimate for all $0 \leq q \leq 1$.

(2) Prove that Steinhaus random Euler products $F_P(1/2+ih) := \prod_{p \leq P} (1 - \frac{f(p)}{p^{1/2+ih}})^{-1}$ satisfy "translation invariance in law", in other words that the joint distribution of the Euler products $(F_P(1/2+ih+it))_{|h| \leq 1/2}$ is the same for all shifts $t \in \mathbb{R}$.

What happens in the Rademacher case?

(3) (A bit harder) Using translation invariance in law, prove that if $2/3 \leq q \leq 1$ then

$$\mathbb{E} \left( \int_{-\infty}^{\infty} \frac{|F_P(1/2+it)|^2}{|1/2+it|^2} dt \right)^q \asymp \mathbb{E} \left( \int_{-1/2}^{1/2} |F_P(1/2+it)|^2 dt \right)^q .$$

## 4. Smooth and Rough Numbers

Recall that a number $n$ is said to be $P$-*smooth* if all prime factors of $n$ are $\leq P$, and to be $P$-*rough* if all prime factors of $n$ are $> P$. Assume that $P \leq y$ is large.

(1) (A bit harder) We can upper bound

$$\sum_{\substack{n \leq y, \\ P \text{ smooth}, \\ \text{squarefree}}} 1 \leq \sqrt{y} + \frac{2}{\log y} \sum_{\substack{\sqrt{y} < n \leq y, \\ P \text{ smooth}, \\ \text{squarefree}}} \log n \quad = \quad \sqrt{y} + \frac{2}{\log y} \sum_{\substack{\sqrt{y} < n \leq y, \\ P \text{ smooth}, \\ \text{squarefree}}} \sum_{p \mid n} \log p$$

$$\ll \quad \sqrt{y} + \frac{e^{-(\log y)/\log P}}{\log y} \sum_{\substack{\sqrt{y} < n \leq y, \\ P \text{ smooth}, \\ \text{squarefree}}} n^{2/\log P} \sum_{p \mid n} \log p.$$

(The last line is a version of *Rankin's trick*.)

By rewriting the double sum and using Chebychev's estimate for $\sum_{p \leq x} \log p$, show that the right hand side is

$$\ll \sqrt{y} + \frac{y e^{-(\log y)/\log P}}{\log y} \sum_{\substack{m \leq y, \\ P \text{ smooth}, \\ \text{squarefree}}} \frac{1}{m^{1-2/\log P}}.$$

(2) By upper bounding by an Euler product, prove that $\sum_{\substack{m \leq y, \\ P \text{ smooth}, \\ \text{squarefree}}} \frac{1}{m^{1-2/\log P}} \ll \log P$.

(3) Finally, remove the assumption of squarefreeness, and deduce that $\sum_{\substack{n \leq y, \\ P \text{ smooth}}} 1 \ll y e^{-(\log y)/\log P}$ (as claimed and used in the calculations in Lecture 2).

(If you are interested in the best known results on counting smooth numbers, there are good surveys on the subject by Hildebrand and Tenenbaum and by Granville.)

(4) Using an upper bound sieve estimate, prove that if $r \leq x^{0.1}$ (say) then

$$\sum_{\substack{x/(r+1) < m \leq x/r, \\ P \text{ rough}}} 1 \ll \prod_{p \leq P} \left(1 - \frac{1}{p}\right) \sum_{x/(r+1) < m \leq x/r} 1 \ll \frac{1}{\log P} \int_{x/(r+1)}^{x/r} 1 dt.$$

(Any standard sieve, e.g. the Selberg sieve, should suffice here— Google or see e.g. Chapter 3 of Montgomery and Vaughan, *Multiplicative Number Theory I*.)

What is important about the size of $r$ to make this work?

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