EXERCISE SHEET FOR MONTRÉAL SUMMER SCHOOL COURSE "INTRODUCTION TO RANDOM MULTIPLICATIVE FUNCTIONS", MAY 2026

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These exercises will be updated as the course goes on, this is *version 1*. Some of the questions will (hopefully!) only take a few minutes, others may require more thought.

1. Orthogonality

(1) Prove that if f is a Rademacher random multiplicative function, and $n, m \in \mathbb{N}$, then
$$\mathbb{E}f(n)f(m) = \begin{cases} \mathbb{E}f(n)^2 = \mathbf{1}_{n \text{ is squarefree}} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$
(Here $\mathbf{1}$ denotes the indicator function.)

(2) Determine the size of the second moment $\mathbb{E}|\sum_{n \leq x} f(n)|^2$, where f is a Rademacher random multiplicative function.

(3) (A bit harder) Find an expression for $\mathbb{E}f^*(n)f^*(m)$, where f^* is an extended Rademacher random multiplicative function.

(4) If f^* is an extended Rademacher random multiplicative function, then it turns out (very roughly) that
$$\sum_{n \leq x} f^*(n) \approx \sqrt{x} \sum_{n \leq x} \frac{f(n)}{\sqrt{n}},$$
where f is a Rademacher random multiplicative function. Prove a precise statement of this kind. What does this suggest about any similarities/differences in the behaviour of f^* and of f?

(5) (Harder) Using total multiplicativity and orthogonality, give the best upper bound you can for $\mathbb{E}|\sum_{n \leq x} f(n)|^{2q}$, where $q \in \mathbb{N}$ and f is a Steinhaus random multiplicative function. You should be able to get a bound of the form $x^q \log^{C(q)} x$, and will probably need to use some information about divisor functions.

If you are interested, see `https://arxiv.org/abs/1804.04114` for the real truth in this problem.

Date: 10th May 2026.

2. Conditioning

(1) Let P be a parameter, and let $\tilde{\mathbb{E}}$ denote expectation conditional on all the values $(f(p))_{p \le P}$. Prove that if f is a Steinhaus random multiplicative function, and a_n are any complex numbers, then

$$\tilde{\mathbb{E}} \sum_{n \le x} a_n f(n) = \sum_{\substack{n \le x, \\ P(n) \le P}} a_n f(n), \qquad \text{and} \qquad \tilde{\mathbb{E}} \left| \sum_{n \le x} a_n f(n) \right|^2 = \sum_{\substack{m \le x, \\ p|m \Rightarrow p > P}} \left| \sum_{\substack{n \le x/m, \\ P(n) \le P}} a_{nm} f(n) \right|^2 .$$

(2) Explicitly verify the Tower Property of conditional expectations: average the above expressions over all possible values of $(f(p))_{p \le P}$ to confirm that

$$\mathbb{E} \sum_{n \le x} a_n f(n) = a_1, \qquad \text{and} \qquad \mathbb{E} \left| \sum_{n \le x} a_n f(n) \right|^2 = \sum_{n \le x} |a_n|^2 .$$

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