L–FUNCTIONS OVER FINITE FIELDS, PROBLEM SET, MAY 2026

1. Consider the standard test function
$$\phi(x) = \Big(\frac{\sin(\pi v x)}{\pi v x}\Big)^2.$$
Let
$$\alpha = \int_{\mathbb{R}} \widehat{\phi}(y)\widehat{W_G}(y)\,dy.$$
Compute α for the classical compact groups (i.e., $G = O, SO(\text{even}), SO(\text{odd}), \text{USp}$).
2. Let $\mathcal{F}$ be a family of L–functions with $\epsilon_f = -1$ for all $f \in \mathcal{F}$. Let
$$p_m = \frac{1}{|\mathcal{F}|}\#\{f \in \mathcal{F} : \text{ord}_{s=1/2}L(f,s) = m\}.$$
Show that for any $\varepsilon > 0$,
$$p_1 > \frac{1}{2}(3 - \alpha - \varepsilon).$$
3. Suppose that $\mathcal{F}$ is a family of L–functions with $G(\mathcal{F}) = SO(\text{odd})$, and suppose we can compute the one-level density in the family for test functions with Fourier transform in $(-2, 2)$. By using the previous two exercises, show that
$$p_1 > \frac{15}{16} - \varepsilon.$$
4. Suppose a family $\mathcal{F}$ has (full) orthogonal symmetry, and we can compute the one-level density in the family with support of the Fourier transform of the test function in $(-v, v)$. Show that by choosing the test function to be
$$\phi(x) = \Big(\frac{\sin(\pi v x)}{\pi v x}\Big)^2,$$
one obtains the optimal proportion of non-vanishing at the central point in the family.
5. Suppose $\mathcal{F}$ is a family with full orthogonal symmetry and one can compute the one-level density in the family with support $(-2, 2)$. Let p_m be as defined in exercise 2. Show that for any $\varepsilon > 0$,
$$\sum_{m=1}^{\infty} m p_m < 1 + \varepsilon.$$
6. Let $\mathcal{H}_{2g+1}$ denote the set of monic, square-free polynomials of degree $2g+1$ over $\mathbb{F}_q[t]$. Suppose that $n > 2g$. Show that for $D \in \mathcal{H}_{2g+1}$,
$$\sum_{f \in \mathcal{M}_n} \Lambda(f)\chi_D(f) = -\sum_{k=1}^{2g} \sum_{f \in \mathcal{M}_k} \chi_D(f) \sum_{h \in \mathcal{M}_{n-k}} \Lambda(h)\chi_D(h).$$
7. Show that
$$\sum_{D \in \mathcal{M}_{2g+1}} L(1/2, \chi_D) = q^{2g+1}\Big(g(1 - \frac{1}{q}) + 1\Big).$$

(8) During lecture, we discussed the phenomenon of "repulsion of zeros" away from $s = 1/2$ for L–functions with symplectic symmetry. What do you expect happens for families with even orthogonal/ odd orthogonal/ unitary symmetry?

(9) Let $C : y^2 = D(x)$, where D is a monic squarefree polynomial in $\mathbb{F}_q[t]$. Let $C(\mathbb{F}_q)$ denote the number of points on C. Show that

$$\#C(\mathbb{F}_q) = q + 1 + \sum_{x \in \mathbb{F}_q} \chi(D(x)),$$

where χ is the quadratic character over $\mathbb{F}_q$.

(10) Let $D \in \mathcal{H}_{2g+1}$. Prove the functional equation of $\mathcal{L}(u, \chi_D)$; namely, show that

$$\mathcal{L}(u, \chi_D) = (\sqrt{q}u)^{2g} \mathcal{L}\Big(\frac{1}{qu}, \chi_D\Big),$$

without using the functional equation of the hyperelliptic curve $C_D : y^2 = D(x)$. (Hint: use Poisson summation for characters over $\mathbb{F}_q[t]$).