

SYMMETRIES OF SINGULAR FOLIATIONS AND LIE INFINITY ACTIONS

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Gone Fishing 2026, CRM | Université de Montréal

April 23 - 26, 2026



UNIVERSITY OF
ILLINOIS
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Let us start with a very simple question

- Let $W \subseteq \mathbb{C}^d$ be an affine variety and $\mathcal{I}_W \subset \mathbb{C}[x_1, \dots, x_d]$ its ideal. Given a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(W)$ on W

$$\begin{array}{ccc} & \mathfrak{X}_W(\mathbb{C}^d) & \\ & \nearrow \tilde{\rho} & \downarrow \\ \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{X}(W) \end{array}$$

Q. Can we extend it to the ambient space?

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Q. Can we extend it to the ambient space?

A. It is a **weak** action on the singular foliation $\mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ over the ambient space \mathbb{C}^d . Any two such actions are "equivalent".

Introduction

We are interested in: A partition of a manifold by submanifolds of possibly different dimensions (called singular foliations)

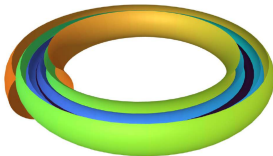


Figure 1: Self-eating-snake singular foliation on $S^1 \times \mathbb{R}^2$.

(Singular) foliations arose from the qualitative study of (partial) differential equations!

Introduction

In differential geometry, Singular foliations arise as

- Lie group actions
- Symplectic leaves of a Poisson manifold
- Orbits of a Lie groupoids or Lie algebroids.
- The list could continue...
- ...

Singular Foliations

Definition (Most used definition)

A **singular foliation** on a smooth, real analytic, or complex manifold M is a subsheaf $\mathcal{F} \subseteq \mathfrak{X}(M)$ of modules over functions that fulfills the following conditions

1. **Stability under Lie bracket:** $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

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1. **Stability under Lie bracket:** $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.
2. **Local finite generateness:** every $m \in M$ admits an open neighborhood $\mathcal{U} \subseteq M$ together with a finite number of vector fields $X_1, \dots, X_r \in \mathfrak{X}(\mathcal{U})$ such that for every open subset $\mathcal{V} \subseteq \mathcal{U}$ the vector fields $X_1|_{\mathcal{V}}, \dots, X_r|_{\mathcal{V}}$ generate \mathcal{F} on \mathcal{V} as a $\mathcal{O}_{\mathcal{V}}$ -module.

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This definition includes the usual definition of (regular) foliation for $\mathcal{F} = \Gamma(F)$, the section of an involutive vector sub-bundle

$$F \subseteq TM.$$

Integration of singular foliations

According to **Hermann's theorem**, a singular foliation induces a partition of the manifold $M = \cup_{\alpha} L_{\alpha}$ into immersed submanifolds $L_{\alpha} \subseteq M$ of possibly "*different*" dimensions, called leaves of the singular foliation.

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In contrast to regular foliations, it remains an "open question" whether a singular foliation can arise from a Lie groupoid

$$\mathcal{G} \rightrightarrows M$$

or as the image of the anchor map of a Lie algebroid.

Singular foliations

The response is **POSITIVE** if we look for "**higher**" Lie groupoid/algebroid structures!

On symmetries of singular foliations

Definition

An **weak symmetry action** of a Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} on M is an \mathbb{K} -linear map $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$, $x \mapsto \underline{x} := \varrho(x)$, that satisfies:

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1. $\forall x \in \mathfrak{g}, [\underline{x}, \mathcal{F}] \subseteq \mathcal{F}$,
2. $\forall x, y \in \mathfrak{g}, \underline{[x, y]}_{\mathfrak{g}} - [\underline{x}, \underline{y}] \in \mathcal{F}$.

\mathfrak{g} does not act!

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Definition

Two weak symmetry actions, $\varrho, \varrho': \mathfrak{g} \rightarrow \mathfrak{X}(M)$, are said to be **equivalent** if there exists a linear map $\varphi: \mathfrak{g} \rightarrow \mathcal{F}$ such that $\varrho - \varrho' = \varphi$.

Examples

Example

- Let $\mathcal{F} := \mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)$ be the singular foliation generated by vector fields vanishing to order 3 at the origin. Let $\mathfrak{g} := \frac{\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)}{\mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)}$. There is a **weak symmetry action** of \mathfrak{g} on \mathcal{F} which assigns to an element in \mathfrak{g} a representative in $\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)$.

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- Let $\pi: M \rightarrow N$ be a submersion with connected fibers. Any Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(N)$ is lifted to a **weak symmetry action** $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ on the regular foliation $\Gamma(\ker d\pi)$, and any two such lifts are equivalent.

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- Any weak action of a Lie algebra \mathfrak{g} on a singular foliation \mathcal{F} on N can be lifted up to a class of **weak symmetry actions** on the pull-back foliation $\pi^{-1}(\mathcal{F})$.

Question

Can a weak symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ on singular foliation (M, \mathcal{F}) be made **strict**? **That is,**

$\forall x, y \in \mathfrak{g}, \varrho'([x, y])_{\mathfrak{g}} - [\varrho'(x), \varrho'(y)] \in \mathcal{F}$ **vanishes**

for some

$$\varrho' \sim \varrho$$

Foliations and Higher Structures

- **Regular case:** a foliation $F \subset TM$ defines a Lie algebroid with injective anchor.

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- Every symmetry $X \in \mathfrak{X}(M)$ of F , i.e., such that $[X, \Gamma(F)] \subset \Gamma(F)$, **lifts** to a symmetry $Z \in \mathfrak{X}(F)$ of (F, d_{dR}) , i.e.,

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Take Z to be the Lie derivative $\mathcal{L}_X \in \mathfrak{X}(F)$.

What happens in the **singular case**?

In general, $\mathcal{F} \subset \mathfrak{X}(M)$ does **NOT** arise as the module of sections of a vector bundle.

Singular foliations & DG-Manifolds

Definition

A **(splitted) dg-manifold** (E, Q) is a \mathbb{N} -graded manifold $E = (M, \Gamma(S(\oplus_{i \in \mathbb{N}} E_{-i}^*)))$ together with a degree +1 **homological** vector field $Q \in \mathfrak{X}_{+1}(E)$, i.e., which satisfies $Q^2 = \frac{1}{2} [Q, Q] = 0$.

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Theorem (Th. Voronov, G. Bonaventurà & N. Poncin)

\mathbb{N} -graded dg-manifolds \simeq Lie ∞ -algebroids

Geometric resolutions

Definition

A submodule of vector fields $\mathcal{F} \subseteq \mathfrak{X}(M)$ is said to be **solvable** if it admits a projective resolution of the form

$$\cdots \longrightarrow \Gamma(E_{-3}) \xrightarrow{d} \Gamma(E_{-2}) \xrightarrow{d} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

also known a **geometric resolution**.

The universal dg-manifold of a SF

Let (M, \mathcal{F}) be a solvable singular foliation.

Theorem (Laurent-Gengoux, Lavau, Strobl, 2019)

Any geometric resolution of \mathcal{F}

$$\dots \xrightarrow{\ell_1} \Gamma(E_{-3}) \xrightarrow{\ell_1} \Gamma(E_{-2}) \xrightarrow{\ell_1} \Gamma(E_{-1}) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

can be given a (unique up to homotopy) Lie ∞ -algebroid structure $(\Gamma(E_\bullet), \ell_\bullet, \rho)$ whose unary bracket is ℓ_1 . It is called the universal Lie ∞ -algebroid of \mathcal{F} .

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Losses	Many more structures for \mathcal{F}
Gains	Projectives modules, free modules, defined on generators

What the theorem says is

Universal Lie ∞ -algebroids \simeq Singular foliations

In my PhD, J. Algebra, 2022: This result is valid even for non-solvable singular foliations!

Results

Theorem (PhD, Journal of Geom. & Phys.)

Every weak symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of \mathfrak{g} on \mathcal{F} “lifts” to a (unique up to homotopy) Lie ∞ -morphism of DGLAs

$$\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$$

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The 2nd Taylor coefficient defines an obstruction class of extending to a strict action!

There are examples where this class is non-zero.

Lift of symmetries (Barcelona 2022, textbook)

Lemma

Any symmetry $X \in \mathfrak{X}(M)$ of the singular foliation \mathcal{F} lifts to a degree zero vector field $Z \in \mathfrak{X}_0(E, Q)$ such that $[Z, Q] = 0$.

Corollary (HARD!)

$$[X, \mathcal{F}] \subseteq \mathcal{F} \implies (\phi_t^X)_* (\mathcal{F}) = \mathcal{F}$$

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Proof. For any admissible t , the flow $\Phi_t^Z : \mathcal{E} \rightarrow \mathcal{E}$ of Z induces an isomorphism of vector bundles $E_{-1} \rightarrow E_{-1}$ such that,

$$\begin{array}{ccc} \Gamma(E_{-1}) & \xrightarrow{(\Phi_t^Z)^{(0)}} & \Gamma(E_{-1}) \\ \downarrow \rho & \circlearrowleft & \downarrow \rho \\ \mathfrak{X}(M) & \xrightarrow{(\varphi_t^X)_*} & \mathfrak{X}(M) \end{array}$$

On weak and strict symmetries: an obstruction theory

Let $\eta: \wedge^2 \mathfrak{g} \rightarrow \Gamma(E_{-1})$ be such that

$$\underline{[x, y]}_{\mathfrak{g}} - [\underline{x}, \underline{y}] = \rho(\eta(x, y)) \in \mathcal{F}.$$

Proposition (Simplified)

Let $m \in M$ be a point of M which is a leaf. Assume that the isotropy Lie algebra \mathfrak{g}_m of \mathcal{F} at m is Abelian. Then, for any weak symmetry action ϱ of a Lie algebra action \mathfrak{g} on \mathcal{F}

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1. \mathfrak{g}_m is a \mathfrak{g} -module.

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1. \mathfrak{g}_m is a \mathfrak{g} -module.
2. The bilinear map, $\eta|_m: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}_m$, is a Chevalley-Eilenberg 2-cocycle of \mathfrak{g} valued in \mathfrak{g}_m .

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3. Its class $cl(\eta) \in H^2(\mathfrak{g}, \mathfrak{g}_m)$ does not depend on the choices made in the construction.

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Let $m \in M$ be a point of M which is a leaf. Assume that the isotropy Lie algebra \mathfrak{g}_m of \mathcal{F} at m is Abelian. Then, for any weak symmetry action ϱ of a Lie algebra action \mathfrak{g} on \mathcal{F}

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3. Its class $cl(\eta) \in H^2(\mathfrak{g}, \mathfrak{g}_m)$ does not depend on the choices made in the construction.
4. Furthermore, $cl(\eta)$ is an obstruction of having a strict symmetry action equivalent to ϱ .

Example

Let $\mathcal{F} := \mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)$. The quotient $\mathfrak{g} := \frac{\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)}{\mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)}$ is an abelian Lie algebra. There is a weak symmetry action of \mathfrak{g} on \mathcal{F} which assigns to an element in \mathfrak{g} a representative in $\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)$.

The theory applies: the class is non-zero.

On weak and strict symmetries on affine varieties: an obstruction theory

Lemma

If a point $p \in W$ is strongly singular i.e., for all $f \in \mathcal{I}_W$, $d_p f \equiv 0$, then the isotropy Lie algebra of the singular foliation

$\mathcal{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ at p is **Abelian**.

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Lemma

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Proposition

Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ be a Lie algebra morphism.

1. Any extension $\tilde{\varrho}: \mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{C}^d)$ is a weak symmetry action for the singular foliation $\mathcal{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$.
2. For any strongly singular point p in W : if the class $cl(\eta)$ does not vanish, the action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ can not be extended to the ambient space.

On weak and strict symmetries: an obstruction theory

Example (Symmetry action on a sub-affine variety)

Let $W \subset \mathbb{C}^2$ be the affine variety generated by the polynomial $\varphi(x, y) = y^2 - x^4$. We consider the vector fields

$$X = (y-x^2) \left(\frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y} \right), Y = (y+x^2) \left(\frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y} \right) \in \mathfrak{X}(W).$$

They satisfy $[X, Y] = 4\varphi \frac{\partial}{\partial y}$.

In this case: the class is non-zero.

On symmetries of a singular foliation

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Thank you for your attention!

References



Camille Laurent-Gengoux and Ruben Louis.

Lie-Rinehart algebras \simeq acyclic Lie ∞ -algebroids.

J. Algebra, 594:1–53, 2022.



Camille Laurent-Gengoux, Sylvain Lavau, and Thomas Strobl.

The universal lie ∞ -algebroid of a singular foliation.

Doc. Math, 25(1571-1652):1806–00475, 2020.



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On symmetries of singular foliations.

Journal of Geometry and Physics, 189:104833, 2023.