Higher Rank KR–modules, Monoidal Categorification and Imaginary Modules

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Overview

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A further link between these subjects is through the notion of monodial categorification introduced by Hernandez and Leclerc.

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This ring has a basis given by the isomorphism classes of irreducible modules. An old result with Pressley says that the index set for the isomorphism classes are given by Drinfeld polynomials. Equivalently, one thinks of this set as monoid on generators $\boldsymbol{\omega}_{i,a}$ where $i \in [1, n]$ and $a \in \mathbb{C}^*$.

A parametrization of irreducible representations

In the case of $GL_n(F)$ the parametrization of irreducible smooth representations (with certain nice properties) is given by the Zelevinsky multi–segments. This is just a collection of intervals $[i_1, j_1], \cdots [i_r, j_r]$ with $i_s, j_s \in \mathbb{Z}$ for all $s \in [1, r]$.

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Each of these intervals determines the character $\det^{(j+i)/2}$ of $GL_{j-i+1}(F)$. The irreducible module determined by the multisegment is the socle of the module for GL_n, $n = \sum_{s=1}^r j_s - i_s$ induced from the parabolic subgroup, $GL_{j_1-i_1+1} \times \cdots \times GL_{j_r-i_r+1}$.

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A translation

It turns out that the two monoids are the same if we impose an integrality condition. Namely we let $\mathcal{P}^+$ be the monoid generated by elements $\boldsymbol{\omega}_{i,a}$ with $i \in [1, n]$, $a \in \mathbb{Z}$ and $a - i \in 2\mathbb{Z}$.

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It turns out now, that it is really more convenient to work with the multi-segment language even for quantum affine algebras.

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If we translate to the language of intervals lets say they are $[i_1, j_1]$ and $[i_2, j_2]$ then this condition just becomes that the intervals overlap, $i_1 < i_2 \leq j_1 < j_2$ or $i_2 < i_1 \leq j_1 <_2$ which is much more pleasant to work with!

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So from now on I am going to use the parametrization of irreducibles in terms of intervals and the elements $\boldsymbol{\omega}_{i,j}$.

Standard modules

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All these results were being proved around 2000 completely independently.

Kirillov–Reshetikhin and Speh modules

Among the best studied modules in the theory of quantum affine algebras are the Kirillov–Reshetikhin modules introduced in 1987. They arose in the context of their work on integrable systems and are associated with the following elements of $\mathcal{P}^+$

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Both families of modules have played a very important role in the independent development of their respective subjects.

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In both case these are modules associated with an ordered multisegment

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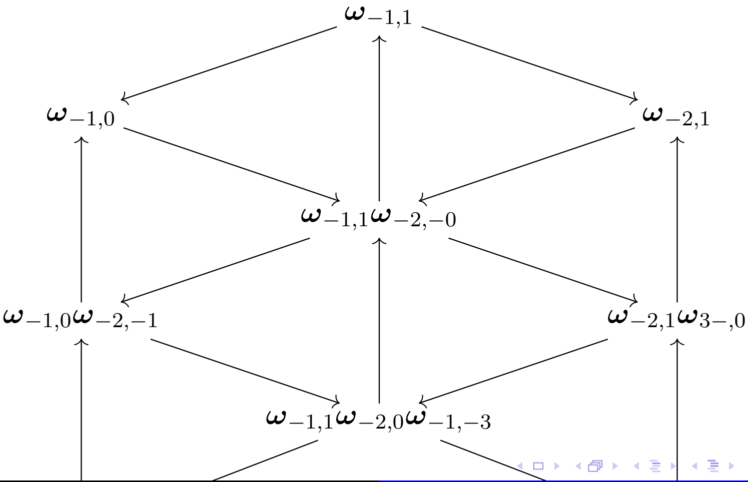
Some of the results they established, by obviously very different methods are identical and one can be deduced from the other by affine Schur Weyl duality.

A new bridge

This came through cluster algebras and the work of Hernandez and Leclerc on monoidal categorification. Consider the following quiver



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Do cluster monomials map to irreducible representations?

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That property is called prime: namely the module cannot be written as a tensor product of two other modules in a non–trivial way.

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That property is called prime: namely the module cannot be written as a tensor product of two other modules in a non–trivial way.

One knows also that any power of a cluster variable is a cluster monomial. So then one also wants the module to be such that any tensor power is irreducible. Such modules are called real.

Prime and Real representations

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A module is called imaginary if its tensor square is reducible. This notion goes back to Leclerc and his counter example to a conjecture of Berenstein–Zelevinsky on dual canonical basis. Leclerc gave a single example in $A_5$of an imaginary modules which proved that such modules existed in A_n, $n \geq 5$.

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All finite dimensional modules of quantum affine A_1 are real.

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Using their conditions one could write down more examples of imaginary modules.

But it is far from being all and since the rank bumps up when using Schur Weyl duality, it left open the question whether imaginary modules existed in A_2 or A_3! It is not easy to generate examples from their restrictions, checking the combinatorial conditions hold is not trivial.

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And both of these came from generalizing the definition of KR–modules.

Generalized KR–modules

Recall that the KR–modules are indexed by elements of the following kind:

$$\boldsymbol{\omega}_{i,j}\boldsymbol{\omega}_{i+1,j+1}\cdots\boldsymbol{\omega}_{i+r,j+r}.$$

We started by asking what would happen if we allowed different kinds of increments in this,

$$\boldsymbol{\omega}_{i,j}\boldsymbol{\omega}_{i+2,j+2},\boldsymbol{\omega}_{i+6,j+6}\cdot\cdot$$

These are more general ladder representations. However if one is not careful with the choice of increments then the module will not be prime.

Higher rank KR–modules

A higher rank KR–module is given by aan element of the form

$$\boldsymbol{\omega}_{i+r_1,j+r_1}\boldsymbol{\omega}_{i+r_2,j+r_2}\cdots\boldsymbol{\omega}_{i+r_\ell,j+r_\ell}$$

where

$$i+r_p < i+r_{p+1} \le j+r_p < j+r_{p+1}, \quad p\in[1,\ell-1].$$

In other words the tensor product of every consecutive pair of fundamental representations is reducible.

The work of Mukhin–Young gives us that these modules are prime, but in this case it is not hard to give a direct proof.

A classification result and a tensor product decomposition

Theorem[Brito-C]

Suppose that we are given an element

$$\boldsymbol{\omega}_{\ell_1,m_1}\cdots\boldsymbol{\omega}_{\ell_r,m_r}\in\mathcal{P}^+$$

with $\ell_1 - m_1 = \cdots = \ell_r - m_r$. Then the corresponding module can be written uniquely as a tensor product of generalized KR–modules.

This theorem is an exact analog of my old result with Pressley for A_1 which was proved in 1990. It is really the first classification result since then and requires a lot of machinery that had been developed in between; for instance the work of [Frenkel-Reshetikhin], [Mukhin-Young].

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The generalized KR–modules are known by the work of [Duan et. al] to be the images of cluster variables in the H-L picture.

Imaginary modules

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Next we tried to understand the tensor product of generalized KR–modules, questions of reducibility, Jordan–Holder series and so on.

Some of this is not known even for KR–modules and this is where we had a nice surprise and recovered Leclerc's example of an imaginary module in a completely different way.

Leclerc's example

Let us consider the following tensor product of KR–modules for A_4.

$$V := V(\boldsymbol{\omega}_{1,3}\boldsymbol{\omega}_{0,2}) \otimes V(\boldsymbol{\omega}_{-1,1}\boldsymbol{\omega}_{-2,0}).$$

Then it is well–known that the trivial module sits inside V; by a result of Kashiwara et al. the trivial is in fact the socle of V. Since we are in small rank it it is not hard to see that there is one more JH–component namely $M := V(\boldsymbol{\omega}_{1,3}\boldsymbol{\omega}_{1,2}\boldsymbol{\omega}_{-1,2}\boldsymbol{\omega}_{-2,0})$.

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And this was precisely the module that Leclerc had shown was imaginary.

But one can now give a very different proof of this. Namely there is a canonical map

$$^{*}V \otimes V \to V \to V(\boldsymbol{\omega}_{1,3}\boldsymbol{\omega}_{0,2}\boldsymbol{\omega}_{-1,1}\boldsymbol{\omega}_{-2,0}) \to 0$$

and the image of $M \otimes M$ is non–zero.

Imaginary modules

Once we understood the example and its proof, the result we wanted was clear, the proof was another matter.

Theorem [Brito-C]

The tensor product of a generalized KR–module with its dual contains an imaginary module whose Drinfeld polynomial can be written explicitly.

These examples in general do not fit into the framework of Lapid–Minguez.

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In our construction, the genralized KR–module and its dual are both cluster variables and the tensor product correspond to the product of the cluster variables.

Since the imaginary module appears in the JH–series, it cannot correspond to any linear combination of cluster monomials.

In other words, this gives an example of a pair of cluster variable whose product is not in the linear span of cluster monomials.