Belavin-Drinfeld Quantum Groups

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Outline

1. Overview
2. Belavin-Drinfeld bialgebras
3. Construction of $\mathcal{O}_{\pi,q}(G)$
4. Construction and description of $U_{\pi,q}(\mathfrak{g})$

Part 1: Overview

Notation: G is a semi-simple complex algebraic group, $\mathfrak{g}$ its Lie algebra; $q \in \mathbb{C}$, generally not a root of unity.

Goal

Extend our understanding of the quantum groups $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$ to the more general Belavin-Drinfeld quantum groups $\mathcal{O}_{\pi,q}(G)$ and $U_{\pi,q}(\mathfrak{g})$.

- Classify the primitive ideals of $\mathcal{O}_{\pi,q}(G)$ and compare this with the description of symplectic leaves in the corresponding Poisson group.
- Construct and describe $U_{\pi,q}(\mathfrak{g})$.
  - Finite dimensional representations
  - Category O for $U_{\pi,q}(\mathfrak{g})$
  - Primitive ideals are annihilators of Verma modules?

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Symplectic leaves and Primitive Spectrum - History

	Quantum Group	Poisson Group	Lie Bialgebra
Standard	$\mathcal{O}_q(G)$ Joseph 1994	(G, r) H-Levasseur 1993	$(\mathfrak{g}, r)$
Multi-parameter	$\mathcal{O}_p(G)$ H-L-Toro 1995	$(G, (r, u))$ H-L-Toro 1995	$(\mathfrak{g}, (r, u))$
Belavin-Drinfeld	$\mathcal{O}_{\pi,q}(G)$ (E-S-S 2000)	$(G, (\pi, u))$ Yakimov 2002	$(\mathfrak{g}, (\pi, u))$ (Bel-Drin 1984)

Description of Symplectic leaves and Primitive Spectrum

The standard case. (H is a maximal torus, W is the Weyl group)

Theorem (H-Levasseur)

1) $Symp\ G = \bigsqcup_{w \in W \times W} Symp_w G$.
2) For each $w \in W \times W$, $Symp_w G$ *is a nonempty* H*-orbit. If* $\mathcal{A}_{\dot{w}}$ *is a fixed symplectic leaf of type* w*, then* $H/Stab\ _H\mathcal{A}_{\dot{w}}$ *is a torus of rank* $rk\ G - s(w)$.
3) The dimension of a leaf of type w *is* $l(w) + s(w)$.

Theorem (Joseph)

1) $Prim\ \mathcal{O}_q(G) = \bigsqcup_{w \in W \times W} Prim_w \mathcal{O}_q(G)$.
2) For each $w \in W \times W$, $Prim_w \mathcal{O}_q(G)$ *is a nonempty* H*-orbit. If* $P_{\dot{w}}$ *is a fixed primitive ideal of type* w*, then* $H/Stab\ _H P_{\dot{w}}$ *is a torus of rank* $rk\ G - s(w)$.

Part 2: Belavin-Drinfeld Lie bialgebra structures

We begin with a brief discussion of the Lie bialgebra structures classified by Belavin and Drinfeld.

	Quantum Group	Poisson Group	Lie Bialgebra
Standard	$\mathcal{O}_q(G)$	(G, r)	$(\mathfrak{g}, r)$
Multi-parameter	$\mathcal{O}_p(G)$	$(G, (r, u))$	$(\mathfrak{g}, (r, u))$
Belavin-Drinfeld	$\mathcal{O}_{\pi,q}(G)$	$(G, (\pi, u))$	$(\mathfrak{g}, (\pi, u))$

Factorizable Coboundary Lie Bialgebras

For $r \in \mathfrak{g} \otimes \mathfrak{g}$, and $X \in \mathfrak{g}$, define $\delta_r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ by $\delta_r(X) = X.r$.

Theorem

Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then δ_r is a cocommutator for $\mathfrak{g}$ if and only if

1. $r + r_{21} \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$.
2. $CYB(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$.

Thus if $\mathfrak{g}$ is a Lie algebra, then $(\mathfrak{g}, \delta_r)$ defines a Lie bialgebra if and only if the two conditions above are satisfies.

Definition

A *quasi-triangular* Lie bialgebra is a coboundary Lie algebra $(\mathfrak{g}, r)$ where $\mathrm{CYB}(r) = 0$. A *quasi-triangular* Lie bialgebra is factorizable if $r + r_{21}$ is non-degenerate.

BD Classification of factorizable Lie bialgebra structures

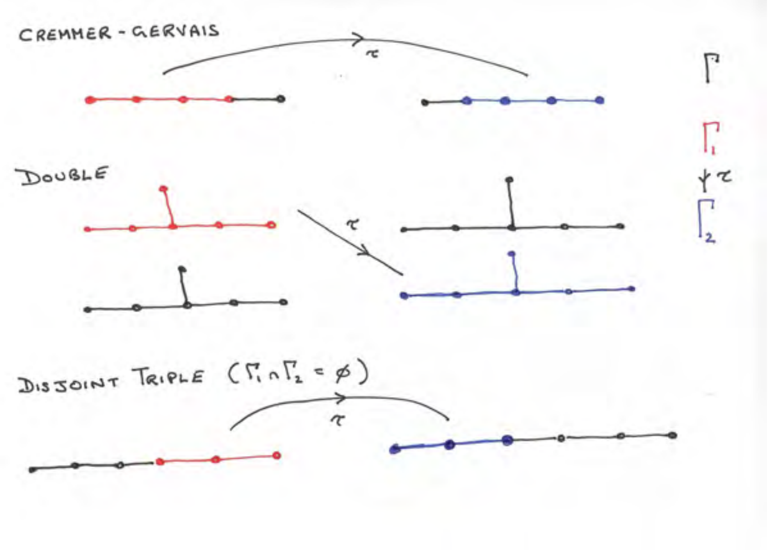
Belavin and Drinfeld classified the Lie bialgebra structures on $\mathfrak{g}$ that are given by factorizable solutions of the CYBE.These Lie bialgebra structures on $\mathfrak{g}$ are given by pairs (τ, s) where τ is a BD triple and $s \in \wedge^2 \mathfrak{h}$ is "compatible" with τ.

Definition

A *Belavin–Drinfeld triple* for $\mathfrak{g}$ is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_i \subset \Gamma$ and $\tau \colon \Gamma_1 \to \Gamma_2$ satisfies

1. $(\tau\alpha, \tau\beta) = (\alpha, \beta)$ for all $\alpha, \beta \in \Gamma_1$ (i.e., τ is a graph isomorphism)
2. for all $\alpha \in \Gamma_1$, there exists a k such that $\tau^k \alpha \notin \Gamma_1$

Examples of Belavin-Drinfeld Triples



The standard classical r-matrix for $\mathfrak{g}$

$$r := \frac{1}{2}\Omega_0 + \sum_{\alpha \in \Delta^+} e_\alpha \otimes f_\alpha.$$

where $\Omega_0 \in S^2\mathfrak{h}$ is the Cartan component of the quadratic Casimir of $\mathfrak{g}$

Theorem (Belavin–Drinfeld, 1984)

The orbits of the adjoint group of $\mathfrak{g}$ on the set of factorizable classical r-matrices for $\mathfrak{g}$ have a unique representative of the form

$$r_{\tau,s} = r - s + \sum_{\alpha \in \Delta^+} \sum_{k=1}^{n} \tau^k(e_\alpha) \wedge f_\alpha$$

for a Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$ of order n and an element

$s \in \wedge^2\mathfrak{h}$ *such that* $2((\alpha_i - \alpha_{\tau(i)}) \otimes 1)s = ((\alpha_i + \alpha_{\tau(i)}) \otimes 1)\Omega_0, \forall i \in \Gamma_1.$

The Dual of Factorizable Quasi-triangular Lie Bialgebra

Let $(\mathfrak{g}, r)$ be a factorizable quasi-triangular Lie bialgebra. Let $r = \sum r_i \otimes r_i'$. Define $\phi_\pm : \mathfrak{g}^* \to \mathfrak{g}$ by

$$\phi_+(\xi) = \sum \xi(r_i) r_i', \quad \phi_-(\xi) = -\sum \xi(r_i') r_i$$

These are Lie bialgebra maps,

$$\phi_\pm : (\mathfrak{g}^*)^{\mathrm{cop}} \to \mathfrak{g}.$$

Let $\mathfrak{p}_\pm = \operatorname{Im} \phi_\pm$ and let $\mathfrak{l} = \mathfrak{p}_+ \cap \mathfrak{p}_-$. All three of $\mathfrak{p}_+$, $\mathfrak{p}_-$ and $\mathfrak{l}$ are Lie subbialgebras of $\mathfrak{g}$. Let $\mathfrak{k} = \ker \phi_+ + \ker \phi_-$.

Theorem (H)

The Lie bialgebra $\mathfrak{g}^*/\mathfrak{k} \cong \mathfrak{l}^*$ is factorizable.

Example

In the standard case $\mathfrak{p}^\pm = \mathfrak{b}^\pm$, $\mathfrak{l} = \mathfrak{h}$, and $\mathfrak{g}^*/\mathfrak{k} \cong \mathfrak{h}^*$ is abelian.

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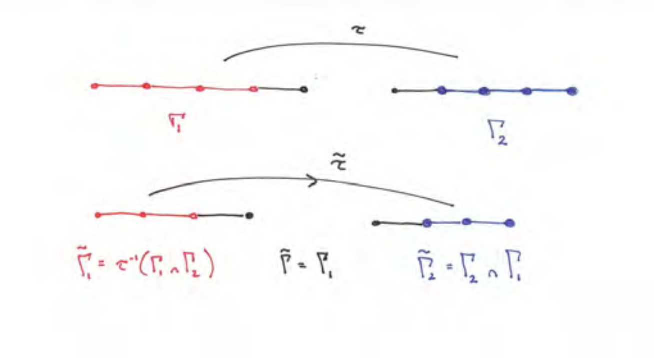
Reduced Triples

Given any triple $(\Gamma_1, \Gamma_2, \tau)$ we have a reduced triple $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\tau})$ given by restricting τ to Γ_1. That is,

$$\tilde{\Gamma} = \Gamma_1, \quad \tilde{\Gamma}_1 = \tau^{-1}(\Gamma_1 \cap \Gamma_2), \quad \tilde{\Gamma}_2 = \Gamma_1 \cap \Gamma_2,$$

Example

The Cremmer-Gervais triple on A_5 and its derived triple on A_4:



We can now precisely describe the structure of in terms of that of $(\mathfrak{g}, f)$.

Theorem (H)

Let (τ, s) be a Belavin-Drinfeld pair and let $(\mathfrak{g}, r)$ be the associated factorizable Lie bialgebra. Then $\tilde{\mathfrak{g}} = \mathfrak{g}^*/\mathfrak{k} \cong \mathfrak{l}_0^*$ is a reductive Lie algebra of type Γ_1. The Lie bialgebra structure on $\tilde{\mathfrak{g}}$ is given by the reduced triple $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\tau})$ and a naturally induced $\tilde{s} \in \wedge^2 \tilde{h}$.

Example

The 'Cremmer-Gervais' bialgebra structure on $\mathfrak{sl}(n+1)$. Set $\mathfrak{g} = \mathfrak{sl}(n+1)$, $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$, $\Gamma_1 = \{\alpha_1, \ldots, \alpha_{n-1}\}, \Gamma_2 = \{\alpha_2, \ldots, \alpha_n\}$ and $\tau(\alpha_i) = \alpha_{i+1}$. Then $\mathfrak{g}^*/\mathfrak{k} \cong \mathfrak{gl}(n)$, and the associated triple is the map $\tilde{\tau} : \{\alpha_1, \ldots, \alpha_{n-2}\} \to \{\alpha_2, \ldots, \alpha_{n-1}\}$ given above.

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Part 3: Construction of $\mathcal{O}_{\pi,q}(G)$

We construct $\mathcal{O}_{\pi,q}(G)$ from $\mathcal{O}_q(G)$ using a 2-cocycle twist.

A *2-cocycle* on a Hopf algebra A is an invertible pairing $\sigma : A \otimes A \to k$ such that $\sigma(1,1) = 1$ and for all x, y and z in A,

$$\sum \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sum \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)})$$

A *braiding* (or dual quasi-triangular structure) on a Hopf algebra A is a bilinear pairing $\beta(\ , \ )$ such that for all a, b, and c in A

1. $\sum \beta(a_{(1)}, b_{(1)})b_{(2)}a_{(2)} = \sum \beta(a_{(2)}, b_{(2)})a_{(1)}b_{(1)}$
2. $\beta(\ , \ )$ is invertible in $(A \otimes A)^*$;
3. $\beta(a, bc) = \sum \beta(a_{(1)}, b)\beta(a_{(2)}, c)$.
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Twisting Braided Hopf Algebras

Theorem

Let A be a braided Hopf algebra with braiding β. Let σ be a 2-cocycle on A. One can twist the multiplication on A to get a new Hopf algebra A_σ. The new multiplication is given by

$$x \cdot y = \sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} \sigma^{-1}(x_{(3)}, y_{(3)}).$$

Moreover $\beta_\sigma = \sigma_{21} * \beta * \sigma^{-1}$ (convolution product) is a braiding on A_σ. The categories of comodules over A and A_σ are equivalent as rigid braided monoidal categories.

Standard Quantum Groups

- Direct construction of $U_q(\mathfrak{g})$ using generators and relations using data from $\mathfrak{g}$.
- Classification of finite dimensional representations of $U_q(\mathfrak{g})$
- Construction of $\mathcal{O}_q(G)$ as restricted dual

For $v \in V$, $\xi \in V^*$,

$$c_{\xi,v}(x) = \xi(xv), x \in U_q(\mathfrak{g}), \quad C(V) = \{c_{\xi,v} \mid \xi \in V^*, v \in V\} \subset U(\mathfrak{g})^o$$

Definition

$$\mathcal{O}_q(G) := \bigoplus_{V \text{simple}} C(V)$$

The standard quantum R-matrix and the $\mathcal{J}$ operators

For $\nu \in \mathcal{Q}^+$, set $m(\nu) := \dim U_q(\mathfrak{n}^+)_\nu = \dim U_q(\mathfrak{n}^-)_{-\nu}$. Let $\{x_{\nu,j}\}_{j=1}^{m(\nu)}$ and $\{x_{-\nu,j}\}_{j=1}^{m(\nu)}$ be two dual bases of $U_q(\mathfrak{n}^+)_\nu$ and $U_q(\mathfrak{n}^-)_{-\nu}$ with respect to the Rosso–Tanisaki form. Define the quasi R-matrix

$$\Theta = \sum_{\nu \in \mathcal{Q}^+} \sum_{j=1}^{m(\nu)} x_{\nu,j} \otimes x_{-\nu,j} \in U_q(\mathfrak{n}^+) \widehat{\otimes} U_q(\mathfrak{n}^-)$$

For $V_1, V_2 \in \mathcal{C}$ and $k \in [1,n]$, the operator $(\tau^k \otimes 1)(\Theta)$ acts on $V_1 \otimes V_2$ because all but finitely many terms in the summation (17) act by 0 on $V_1 \otimes V_2$. Define $\mathcal{J}_{V_1,V_2} \in \mathrm{End}_{\mathbb{C}}(V_1 \otimes V_2)$ by

$$\mathcal{J}_{V_1,V_2} := q^{(\tau\otimes 1)\Omega_0}((\tau \otimes 1)\Theta) \ldots q^{(\tau^n\otimes 1)\Omega_0}((\tau^n \otimes 1)\Theta) \times q^{-(\tau^n\otimes 1)\Omega_0 - \cdots - (\tau\otimes 1)\Omega_0} q^{-s_0 - \Omega_{\mathfrak{z}^\perp}/2}.$$

Theorem (Etingof-Schedler-Schiffman)

For all $V_1, V_2, V_3 \in \mathcal{C}$, the $\mathcal{J}$-operators satisfy the 2-cocycle condition

$$\mathcal{J}_{V_1 \otimes V_2, V_3} \mathcal{J}_{V_1,V_2} = \mathcal{J}_{V_1, V_2 \otimes V_3} \mathcal{J}_{V_2,V_3}.$$

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Theorem

The map $\theta_\pi : \mathcal{O}_q(G) \times \mathcal{O}_q(G) \to \mathbb{C}$ *defined by*

$$\theta_\pi(c_{\xi_1,x_1}, c_{\xi_2,x_2}) := \langle \xi_1 \otimes \xi_2, \mathcal{J}_{V_1,V_2}(v_1 \otimes v_2)\rangle$$

is a Hopf algebra 2-cocycle on $\mathcal{O}_q(G)$*.*

Denote the twisted Hopf algebra

$$\mathcal{O}_{\pi,q}(G) := \mathcal{O}_q(G)_{\theta_\tau}.$$

$$(c_{\xi_1,x_1} c_{\xi_2,x_2})(x) = \langle \xi_1 \otimes \xi_2, \mathcal{J}^{-1}_{V_1,V_2} \Delta(x) \mathcal{J}_{V_1,V_2}(v_1 \otimes v_2)\rangle$$

Example

- The Cremmer-Gervais quantum groups $\mathcal{O}_{CG,q}(SL(n))$ (constructed using an explicit R-matrix)
- The double quantum groups $\mathcal{O}_q(D(G)) \cong \mathcal{O}_q(G) \bowtie \mathcal{O}_q(G)$ (constructed using the Drinfeld double construction)

Part 4: The FRT dual $U_{\pi,q}(\mathfrak{g})$

For a braided Hopf algebra (A, β) we have Hopf algebra maps $l^{\pm} : A^{op} \to A^{\circ}$ by

$$l^{+}(a)(b) = \beta(a, S(b)) \quad \text{and} \quad l^{-}(a)(b) = \beta(b, S(a))$$

Let $U^{\pm} = l^{\pm}(A)$. The *FRT dual* of A is the Hopf subalgebra $U(A)$ generated by U^{+} and U^{-}.

The braiding on A induces a Hopf pairing on $U^{+} \otimes (U^{-})^{op}$ using which one can construct the Drinfeld double $U^{+} \bowtie U^{-}$. The multiplication map $u \otimes v \mapsto uv$ is then a surjective Hopf algebra map

$$\mu : U^{+} \bowtie U^{-} \to U(A)$$

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The pairing between $\mathcal{O}_{\pi,q}(G)$ and $U_{\pi,q}(\mathfrak{g})$

$\mathcal{O}_{\pi,q}(G)$ has the *same coalgebra* structure as $\mathcal{O}_q(G)$, but a *twisted algebra structure*.

NOT TRUE that $U_{\pi,q}(\mathfrak{g})$ has the *same algebra* structure as $U_q(\mathfrak{g})$, but a *twisted coalgebra structure*.

Heuristically,

$$U_{\pi,q}(\mathfrak{g}) \cong \mathcal{O}_q(G_r)$$

a quantization of the algebra of functions on G_r is the dual Poisson group

Theorem (H-Yakimov)

The pairing between $\mathcal{O}_{\pi,q}(G)$ and $U_{\pi,q}(\mathfrak{g})$ is non-degenerate.

Corollary

$U_{\pi,q}(\mathfrak{g})$ has a category of finite dimensional modules equivalent to the category of finite dimensional weight modules over $U_q(\mathfrak{g})$. $\mathcal{O}_{\pi,q}(G)$ is the restricted dual of $U_{\pi,q}(\mathfrak{g})$ with respect to this category.

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The braiding on $U_0 = U^+ \cap U^-$

Set

$$U_0 = U^+ \cap U^-$$

The pairing between U^+ and U^- induces a *braiding* on U_0

Thus the FRT dual $U(A)$ of a braided Hopf algebra A contains a canonical braided Hopf subalgebra $U_0(A)$.

Problem

Describe U_0 for the Belavin-Drinfeld quantum group $\mathcal{O}_{\pi,q}(G)$.

Example

For the trivial triple we have $U_0(\mathcal{O}_q(G)) = \mathbb{C}[K_i^{\pm 1}] \cong \mathcal{O}(T)$, for T a maximal torus inside G, equipped with the braiding given by the Rosso form.

$$\begin{array}{ccccc} & & \mathfrak{p}^+ & & \\ & \overset{\phi^+}{\nearrow} & & \searrow & \\ \mathfrak{g}^* & & & & \mathfrak{g}^*/\mathfrak{k} \cong \tilde{\mathfrak{g}} \\ & \overset{\phi^-}{\searrow} & & \nearrow & \\ & & \mathfrak{p}^- & & \end{array} \qquad \begin{array}{ccccc} & & \mathfrak{p}^+ & & \\ & \nearrow & & \searrow & \\ \mathfrak{p}^+ \cap \mathfrak{p}^- = \tilde{\mathfrak{g}}^* & & & & \mathfrak{g} \\ & \searrow & & \nearrow & \\ & & \mathfrak{p}^- & & \end{array}$$

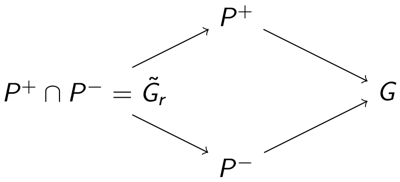
$$\begin{array}{ccccc} & & P^+ & & \\ & \nearrow & & \searrow & \\ G_r & & & & \tilde{G} \\ & \searrow & & \nearrow & \\ & & P^- & & \end{array} \qquad \begin{array}{ccccc} & & P^+ & & \\ & \nearrow & & \searrow & \\ P^+ \cap P^- = \tilde{G}_r & & & & G \\ & \searrow & & \nearrow & \\ & & P^- & & \end{array}$$

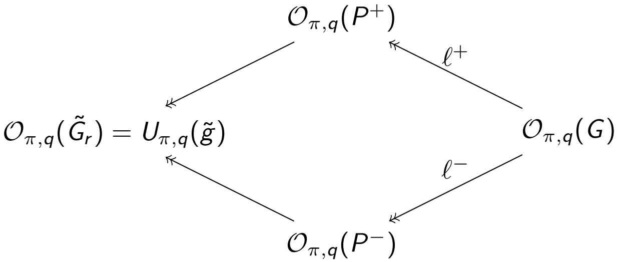
Heuristics I

$$\begin{array}{ccccc} & & P^{+} & & \\ & \nearrow\!\!\!\!\nearrow & & \searrow\!\!\!\!\searrow & \\ G_r & & & & \tilde{G} \\ & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\nearrow & \\ & & P^{-} & & \end{array}$$

$$\begin{array}{ccccc} & & U^{+} = \mathcal{O}_{\pi,q}(P^{+}) & & \\ & \swarrow & & \nwarrow & \\ U_{\pi,q}(\mathfrak{g}) = \mathcal{O}_{\pi,q}(G_r) & & & & U_0 = \mathcal{O}_{\pi,q}(\tilde{G}) \\ & \nwarrow & & \swarrow & \\ & & U^{-} = \mathcal{O}_{\pi,q}(P^{-}) & & \end{array}$$

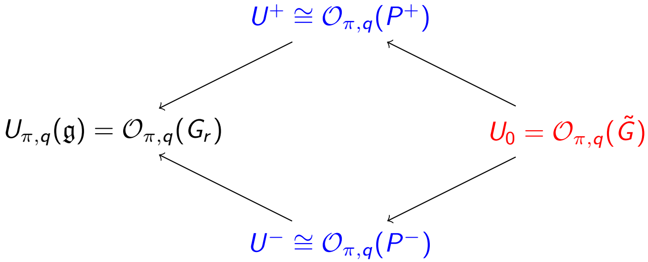
Heuristic II

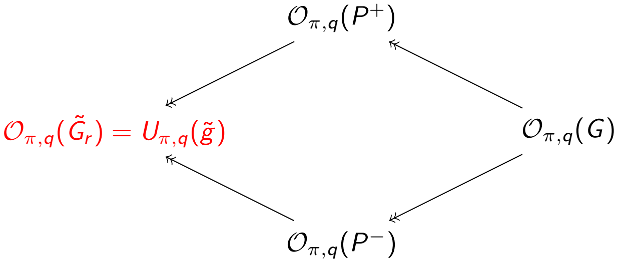




$$\begin{array}{ccccc} & & U_q(\mathfrak{b}^+) \cong \mathcal{O}_q(B^+) & & \\ & \swarrow & & \nwarrow & \\ U_q(\mathfrak{g}) = \mathcal{O}_q(G_r) & & & & U_0 = \mathbb{C}[K_i^{\pm}] \\ & \nwarrow & & \swarrow & \\ & & U_q(\mathfrak{b}^-) \cong \mathcal{O}_q(B^-) & & \end{array}$$

$$\begin{array}{ccccc} & & \mathcal{O}_q(B^+) & & \\ & \swarrow\!\!\swarrow & & \nwarrow\!\!\nwarrow & \\ \mathcal{O}(T) & & & & \mathcal{O}_q(G) \\ & \nwarrow\!\!\nwarrow & & \swarrow\!\!\swarrow & \\ & & \mathcal{O}_q(B^-) & & \end{array}$$


$U^+ \cong \mathcal{O}_{\pi,q}(P^+)$
$U_{\pi,q}(\mathfrak{g}) = \mathcal{O}_{\pi,q}(G_r)$
$U_0 = \mathcal{O}_{\pi,q}(\tilde{G})$
$U^- \cong \mathcal{O}_{\pi,q}(P^-)$


$\mathcal{O}_{\pi,q}(P^+)$
$\mathcal{O}_{\pi,q}(\tilde{G}_r) = U_{\pi,q}(\tilde{\mathfrak{g}})$
$\mathcal{O}_{\pi,q}(G)$
$\mathcal{O}_{\pi,q}(P^-)$

Let

$$C^{\pm} = (U_q(\mathfrak{p}^{+}))^{\perp} = \{c \in \mathcal{O}_{\pi,q}(G) \mid \langle c, U_q(\mathfrak{p}^{\pm})\rangle = 0\},$$

and define

$$\mathcal{O}_{\pi,q}(P^{\pm}) = \mathcal{O}_{\pi,q}(G)/C^{\pm}$$

Theorem (H-Y)

We have $\ker l^{\pm} = C^{\pm}$ *so*

$$U^{\pm} \cong \mathcal{O}_{\pi,q}(P^{\pm})$$

Moreover the map

$$(\ell^{+} \otimes \ell^{-})\Delta \colon \mathcal{O}_{T,q}(G) \to \mathcal{O}_{T,q}(P^{+}) \otimes \mathcal{O}_{T,q}(P^{-}).$$

is an embedding.

Conjecture

There exists a reductive Lie group $\tilde{G}$ *with associated Lie algebra* $\tilde{\mathfrak{g}}$ *for which* $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\tau})$ *is a triple for* $\tilde{\mathfrak{g}}$ *and such that*

$$U_0(\mathcal{O}_{\pi,q}(G)) \cong \mathcal{O}_{\tilde{\pi},q}(\tilde{G})$$

for $\tilde{\pi} = (\tilde{\tau}, \tilde{u})$ *suitable choice of continuous parameter* $\tilde{u}$*.*

Example

- For the trivial triple we have $U_0(\mathcal{O}_q(G)) \cong \mathcal{O}(K)$, for K a maximal torus inside G
- For the Cremmer-Gervais quantum group, it was shown that

$$U_0(\mathcal{O}_{CG,q}(SL(n))) \cong \mathcal{O}_{CG,q,u}(GL(n-1)))$$

- For the double quantum groups, we have

$$U_0(\mathcal{O}_q(D(G)) \cong \mathcal{O}_q(G) \otimes \mathcal{O}(K)$$

Conjecture

Dual to the embedding

$$\mathcal{O}_{\tilde{\pi},q}(\tilde{G}) \hookrightarrow U_{\pi,q}(\mathfrak{g})$$

is a surjective map

$$\mathcal{O}_{\pi,q}(G) \twoheadrightarrow U_{\tilde{\pi},q}(\tilde{g})$$

Problem

Describe the primitive spectrum of $\mathcal{O}_{\pi,q}(G)$

Any such classification would have to include the classification of primitive ideals in $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$!