m-periodic Gorenstein objects

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In M. Auslander, M. Bridger, *Stable module theory*, in: Memoirs. Amer. Math. Soc., vol. 94. American Mathematical Society, Providence, RI, 1969. the authors introduced the G-dimension of a module M (finitely generated) over Noetherian rings.

- An exact complex $L_\bullet$ of projective R-modules is *totally acyclic* if $\mathrm{Hom}(L_\bullet, Q)$ is also exact for any projective R-module Q.

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- An R-module M is said to be *totally reflexive* if there exists a totally acyclic complex $L_\bullet$ of finitely generated projective modules such that $M = \mathrm{Ker}(L_0 \to L_{-1})$.
- Let R be a noetherian ring. A *G-resolution* of a finitely generated R-module M is an exact sequence $\cdots \to G_1 \to G_0 \to M \to 0$ with each G_i a totally reflexive module.

Definition

Let R be a noetherian ring. For a finitely generated R-module $N \neq 0$ the *G-dimension*, denoted $G\text{-dim}_R(N)$, is the least integer $n \geq 0$ such that there exists a G-resolution of N with $G_i = 0$ for all $i > n$. If no such n exists, then $G\text{-dim}_R(N)$ is infinite. By convention, $G\text{-dim}_R(0) = -\infty$.

E. Enochs and O. M. G. Jenda. Gorenstein injective and projective modules. Math. Z. 220:611-633, 1995.

Definition

A module M is *Gorenstein projective* if there exists an exact complex of projective modules $P_\bullet : \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$, such that for any projective R-module Q, the complex $\mathrm{Hom}(P_\bullet, Q)$ is still exact, and such that $M = \mathrm{Ker}(P_0 \to P_{-1})$. We denote the class of such modules by $\mathcal{GP}(R)$.

D. Bennis and N. Mahdou. Strongly Gorenstein projective, injective and flat modules. J. Pure Appl. Algebra 210 (2):437-445, 2007.

Definition

A module M is *strongly Gorenstein projective* if there exists an exact complex $\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$, with P a projective module, such that for any projective R-module Q, the complex $\cdots \to \mathrm{Hom}(P, Q) \to \mathrm{Hom}(P, Q) \to \mathrm{Hom}(P, Q) \to \cdots$ is still exact, and such that $M = \mathrm{Ker}(P \xrightarrow{f} P)$.

A module M is *strongly Gorenstein projective* if there exists a short exact sequence $\eta : 0 \to M \to P \to M \to 0$, with P a projective module, such that for any projective R-module Q, the complex $\mathrm{Hom}(\eta, Q)$ is exact.

D. Bennis and N. Mahdou. A generalization of strongly Gorenstein projective modules. J. Algebra Appl. (8):219-227, 2009.

Definition

Let n be a positive integer. An R-module M is said to be *n-strongly Gorenstein projective*, if there exists an exact sequence of R-modules $0 \to M \to P_n \to \cdots \to P_1 \to M \to 0$, where each P_i is projective, such that $\mathrm{Hom}(-, Q)$ leaves the sequence exact whenever Q is a projective R-module. We denote this class of modules by $n\mathcal{SGP}(R)$.

• 1-strongly Gorenstein projective modules= strongly Gorenstein projective modules.

• strongly Gorenstein projective modules $\subseteq$ n-strongly Gorenstein projective modules $\subseteq$ Gorenstein projective modules

V. Becerril, O. Mendoza and V. Santiago. Relative Gorenstein objects in abelian categories. Comm. Algebra, 49(1):352-402, 2020.

Definition

Let $\mathcal{C}$ be an abelian category and $(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{C}^2$. A *left complete* $(\mathcal{A}, \mathcal{B})$*-resolution* is an exact complex $\eta : \cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$ with $A_i, A^i \in \mathcal{A}$ for all i and such that the complex $\mathrm{Hom}_{\mathcal{C}}(\eta, B)$ is exact for any $B \in \mathcal{B}$. The object $M = \mathrm{Ker}(A_0 \to A^0)$ is called $(\mathcal{A}, \mathcal{B})$-**Gorenstein projective**. We denote the class of such objects by $\mathcal{GP}_{(\mathcal{A},\mathcal{B})}$.

$$n\mathcal{SGP}(R) \subseteq \mathcal{GP}(R)$$

$$? \subseteq \mathcal{GP}_{(\mathcal{A},\mathcal{B})}$$

Definition

Let $(\mathcal{A}, \mathcal{B})$ be a pair of classes of objects in an abelian category $\mathcal{C}$. We say that an object $M \in \mathcal{C}$ is ***m*-periodic $(\mathcal{A}, \mathcal{B})$-Gorenstein projective** if there is an $\mathcal{A}$-loop at M of length m, that is, an exact complex

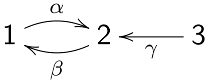
$$\eta : 0 \to M \to A_m \to \cdots \to A_1 \to M \to 0$$

with $A_k \in \mathcal{A}$ for every $1 \leq k \leq m$ which is $\mathrm{Hom}(-, \mathcal{B})$-acyclic. We denote by $\pi\mathcal{GP}_{(\mathcal{A},\mathcal{B},m)}$ the class of m-periodic $(\mathcal{A}, \mathcal{B})$-Gorenstein projective objects in $\mathcal{C}$.

P. Zhang and B.-L. Xiong. Separated monic representations II: Frobenius categories and RSS equivalences. *Trans. Amer. Math. Soc.*, 372(2):981-1021, 2019.

Example:

Let k be a field and Λ be the path algebra over k given by the quiver



with relations $\alpha\beta = 0 = \beta\alpha$.

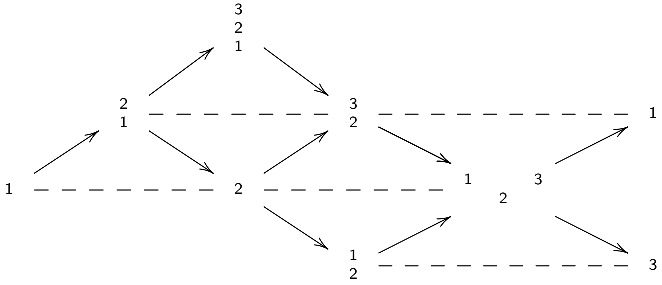
The indecomposable projective Λ-modules are

$$P(1) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \qquad P(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \qquad \text{and} \qquad P(3) = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix},$$

while the indecomposable injective Λ-modules are given by

$$I(1) = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}, \qquad I(2) = \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix}, \qquad \text{and} \qquad I(3) = \begin{smallmatrix} 3 \end{smallmatrix}.$$

On the other hand, the Auslander-Reiten quiver of Λ is



where the vertices i represent the same simple module $S(i)$.

Consider the class $\mathcal{X} = \mathrm{add}(S(1) \oplus P(2) \oplus S(2) \oplus P(1))$ of Λ-modules isomorphic to direct summands of finite direct sums of $S(1) \oplus P(2) \oplus S(2) \oplus P(1)$. Then,

1. $\mathcal{X}$ is closed under extensions.
2. $\mathcal{X}$ is a Frobenius subcategory of $\mathrm{mod}(\Lambda)$ and $\mathcal{P}(\mathcal{X}) = \mathrm{add}(P(1) \oplus P(2))$.

If we set $\mathcal{A} = \mathcal{B} = \mathcal{P}(\mathcal{X})$.

1. $P(3)$ is a 1-strongly Gorenstein projective Λ-module.
2. $P(3) \notin \pi\mathcal{GP}_{(\mathcal{A},\mathcal{B},m)}$ for any $m \geq 1$.

Gorenstein dimensions

D. Bennis & N. Mahdou. Global Gorenstein dimensions. Proc. Am. Math. Soc., 138(2):461-465, 2010.

$$\sup\{\mathsf{Gpd}_R(M) : M \in \mathrm{Mod}(R)\} = \sup\{\mathsf{Gid}_R(M) : M \in \mathrm{Mod}(R)\}$$

Assume first that M is strongly Gorenstein projective. By Remark 2.3 there is a short exact sequence $0 \to M \to P \to M \to 0$ with P is projective. The Horseshoe Lemma (see [10, Remark, page 187]) gives a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & P & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I_0 & \to & I_0 \oplus I_0 & \to & I_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I_n & \to & E_n & \to & I_n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where I_i is injective for $i = 0, ..., n-1$. Since P is projective, $\mathrm{id}_R(P) \leq n$ (by Lemma 2.1); hence E_n is injective. On the other hand, from [7, Theorem 2.2], $\mathrm{pd}_R(E) \leq n$ for every injective R-module E. Then, $\mathrm{Ext}^i_R(E, I_n) = 0$ for all $i \geq n+1$. Then, from Remark 2.3, I_n is strongly Gorenstein injective, and so $\mathrm{Gid}_R(M) \leq n$. This implies, from [6, Proposition 2.19], that $\mathrm{Gid}_R(G) \leq n$ for any Gorenstein projective R-module G, since every Gorenstein projective R-module is a direct summand of a strongly Gorenstein projective R-module (Lemma 2.4).

Finally, consider an R-module M with $\mathrm{Gpd}_R(M) \leq m \leq n$. We can assume that $\mathrm{Gpd}_R(M) \neq 0$. Then, there exists a short exact sequence $0 \to K \to N \to M \to 0$ such that N is Gorenstein projective and $\mathrm{Gpd}_R(K) \leq m-1$ [6, Proposition 2.18]. By induction, $\mathrm{Gid}_R(K) \leq n$ and $\mathrm{Gid}_R(N) \leq n$. Therefore, using [6, Theorems 2.22 and 2.25] and the long exact sequence of Ext, we get that $\mathrm{Gid}_R(M) \leq n$. □

D. Bennis & N. Mahdou. Strongly Gorenstein projective, injective, and flat modules. J. Pure Appl. Algebra, 210(2):437-445, 2007.

A module is Gorenstein projective if, and only if, it is a direct summand of a strongly Gorenstein projective module.

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$. The pair $(\mathcal{A}, \mathcal{B})$ is *GP-admissible* (resp., GI-admissible) if:

- $\mathcal{A}$ and $\mathcal{B}$ are closed under finite coproducts and $\mathrm{pd}_{\mathcal{B}}(\mathcal{A}) = 0$.
- Every object in $\mathcal{C}$ is the epimorphic image of an object in $\mathcal{A}$ (resp., every object in $\mathcal{C}$ can be embedded into an object in $\mathcal{B}$).
- $\mathcal{A}$ (resp., $\mathcal{B}$) is closed under extensions.
- $\mathcal{A} \cap \mathcal{B}$ is a relative cogenerator in $\mathcal{A}$ (resp., a relative generator in $\mathcal{B}$).

D. Bravo, J. Gillespie, M. Hovey, The stable category of a general ring, Preprint, arXiv:1405.5768, 2014.

Definition

A (left) module M over a ring R is said to be of type FP_∞ if M has a projective resolution by finitely generated modules. We denote by $FP_\infty(R)$ the class of such modules.

Examples

GP-admissible pairs: $(\mathcal{P}(R), \mathcal{P}(R))$, $(\mathcal{P}(R), \mathcal{F}(R))$ and $(\mathcal{P}(R), \mathcal{L}(R))$ where:

- $\mathcal{P}(R)$:=projective R-modules.
- $\mathcal{F}(R)$:=flat R-modules.
- $\mathcal{L}(R)$:=level R-modules= $\{N \in \mathrm{Mod}(R) : \mathrm{Tor}_R^1(M, N) = 0 \quad \forall\, M \in \mathrm{FP}_\infty(R^{op})\}$.

Examples

GI-admissible pairs: $(\mathcal{I}(R), \mathcal{I}(R))$, (FP-$\mathcal{I}(R), \mathcal{I}(R))$ and $(\mathrm{FP}_\infty\text{-}\mathcal{I}(R), \mathcal{I}(R))$ where

- $\mathcal{I}(R)$:=injective R-modules.
- FP-$\mathcal{I}(R)$:=absolutely pure R-modules$=\{M \in \mathrm{Mod}(R) : \mathrm{Ext}^1_R(F, M) = 0 \quad \forall\, F$ finit. present.$\}$.
- FP_∞-$\mathcal{I}(R)$:=absolutely clean R-modules$=\{M \in \mathrm{Mod}(R) : \mathrm{Ext}^1_R(F, M) = 0 \quad \forall\, F \in \mathrm{FP}_\infty(R)\}$.

Proposition

Let $(\mathcal{A}, \mathcal{B})$ be a GP-admissible pair in an AB4 abelian category $\mathcal{C}$, with $\mathcal{A}$ closed under coproducts. Then, an object in $\mathcal{C}$ is $(\mathcal{A}, \mathcal{B})$-Gorenstein projective if, and only if, it is a direct summand of a 1-periodic $(\mathcal{A}, \mathcal{B})$-Gorenstein projective object. In other words,

$$\mathcal{GP}_{(\mathcal{A},\mathcal{B})} = \mathrm{add}(\pi\mathcal{GP}_{(\mathcal{A},\mathcal{B},1)}).$$

In what follows, given an abelian category $\mathcal{C}$, $(\mathcal{A}, \mathcal{B})$ will be a GP-admissible pair, $(\mathcal{Z}, \mathcal{W})$ will be a GI-admissible pair in $\mathcal{C}$. We shall write $\omega := \mathcal{A} \cap \mathcal{B}$ and $\nu := \mathcal{Z} \cap \mathcal{W}$.

Suppose that the following conditions hold:

1. ω and ν are closed under direct summands.
2. $\mathrm{Ext}^1(\mathcal{GP}_{(\mathcal{A},\mathcal{B})}, \nu) = 0$ and $\mathrm{Ext}^1(\omega, \mathcal{GI}_{(\mathcal{Z},\mathcal{W})}) = 0$.
3. Every object in $\mathcal{GP}_{(\mathcal{A},\mathcal{B})}$ admits a $\mathrm{Hom}(-,\nu)$-acyclic ν-coresolution, and every object in $\mathcal{GI}_{(\mathcal{Z},\mathcal{W})}$ admits a $\mathrm{Hom}(\omega,-)$-acyclic ω-resolution.

Given $C \in \mathcal{C}$, the $(\mathcal{A}, \mathcal{B})$-Gorenstein projective dimension of C, denoted $\mathrm{Gpd}_{(\mathcal{A},\mathcal{B})}(C)$ is defined as the resolution dimension

$$\mathrm{Gpd}_{(\mathcal{A},\mathcal{B})}(C) = \mathrm{resdim}_{\mathcal{GP}_{(\mathcal{A},\mathcal{B})}}(C).$$

Similarly, $\mathrm{Gid}_{(\mathcal{Z},\mathcal{W})}(C) = \mathrm{coresdim}_{\mathcal{GI}_{(\mathcal{Z},\mathcal{W})}}(C)$ denotes and defines the $(\mathcal{Z}, \mathcal{W})$-Gorenstein injective dimension of C.

If $\mathcal{X}$ is a class of objects in $\mathcal{C}$, the $(\mathcal{A}, \mathcal{B})$-Gorenstein projective dimension and $(\mathcal{Z}, \mathcal{W})$-Gorenstein injective dimension of $\mathcal{X}$ are defined as

$$\mathrm{Gpd}_{(\mathcal{A},\mathcal{B})}(\mathcal{X}) := \mathrm{resdim}_{\mathcal{GP}_{(\mathcal{A},\mathcal{B})}}(\mathcal{X}),$$

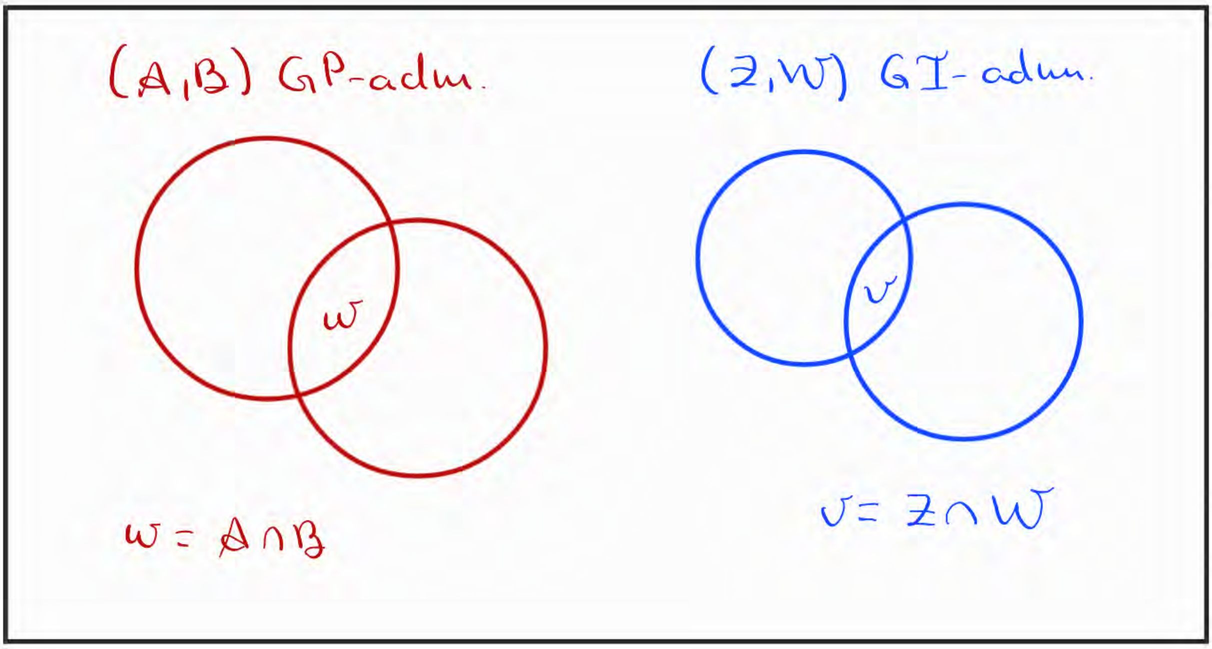
$$\mathrm{Gid}_{(\mathcal{Z},\mathcal{W})}(\mathcal{X}) := \mathrm{coresdim}_{\mathcal{GI}_{(\mathcal{Z},\mathcal{W})}}(\mathcal{X}).$$

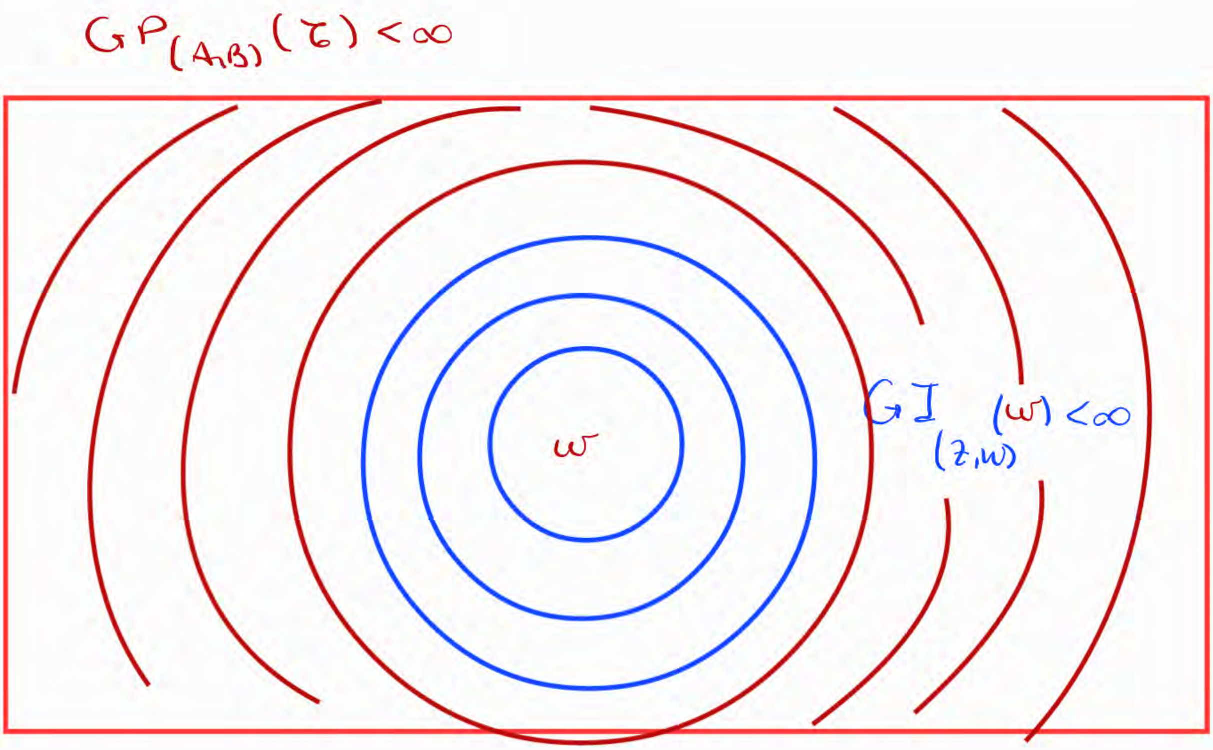
The global $(\mathcal{A}, \mathcal{B})$-Gorenstein projective and global $(\mathcal{Z}, \mathcal{W})$-Gorenstein injective dimensions are defined as $\mathrm{gl.GPD}_{(\mathcal{A},\mathcal{B})}(\mathcal{C}) := \mathrm{Gpd}_{(\mathcal{A},\mathcal{B})}(\mathcal{C})$ and $\mathrm{gl.GID}_{(\mathcal{Z},\mathcal{W})}(\mathcal{C}) := \mathrm{Gid}_{(\mathcal{Z},\mathcal{W})}(\mathcal{C})$.

Theorem

If $\mathrm{gl.GPD}_{(\mathcal{A},\mathcal{B})}(\mathcal{C}) < \infty$ *and* $\mathrm{Gid}_{(\mathcal{Z},\mathcal{W})}(\omega) < \infty$ *then*

$$\mathrm{gl.GID}_{(\mathcal{Z},\mathcal{W})}(\mathcal{C}) = \mathrm{gl.GPD}_{(\mathcal{A},\mathcal{B})}(\mathcal{C}).$$


(A,B) GP-adm.
(Z,W) GI-adm.
w
v
w = A ∩ B
v = Z ∩ W


GP(A,B)(τ) < ∞
w
GI(Z,W)(w) < ∞

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Introduction

m-periodic Gorenstein objects

Gorenstein dimensions

GP-admissible pairs

We denote the global Gorenstein dimension of the previous pairs as follows:

1. $\mathrm{glGPD}(R) := \mathrm{Gpd}_{(\mathcal{P}(R),\mathcal{P}(R))}(\mathrm{Mod}(R))$.
2. $\mathrm{glDPD}(R) := \mathrm{Gpd}_{(\mathcal{P}(R),\mathcal{F}(R))}(\mathrm{Mod}(R))$.
3. $\mathrm{glGPD}_{AC}(R) := \mathrm{Gpd}_{(\mathcal{P}(R),\mathcal{L}(R))}(\mathrm{Mod}(R))$.
4. $\mathrm{glGID}(R) := \mathrm{Gid}_{(\mathcal{I}(R),\mathcal{I}(R))}(\mathrm{Mod}(R))$.
5. $\mathrm{glDID}(R) := \mathrm{Gid}_{(\mathrm{FP}\text{-}\mathcal{I}(R),\mathcal{I}(R))}(\mathrm{Mod}(R))$.
6. $\mathrm{glGID}_{AC}(R) := \mathrm{Gid}_{(\mathrm{FP}_{\infty}\text{-}\mathcal{I}(R),\mathcal{I}(R))}(\mathrm{Mod}(R))$.

- For any arbitrary ring, ④ = ① = ②.
- For any Ding-Chen ring, ① = ④ = ② = ⑤.
- For any commutative ring, ③ = ⑥.

Corollary

The following equality holds for any arbitrary ring R:

$$①=②=③=④=⑤=⑥$$

Proof. Note that the containments $\mathcal{P}(R) \subseteq \mathcal{F}(R) \subseteq \mathcal{L}(R)$ and $\mathcal{I}(R) \subseteq \text{FP-}\mathcal{I}(R) \subseteq \text{FP}_\infty\text{-}\mathcal{I}(R)$ imply that

$$① \leq ② \leq ③ \quad \text{and} \quad ④ \leq ⑤ \leq ⑥.$$

Proof. Note that the containments $\mathcal{P}(R) \subseteq \mathcal{F}(R) \subseteq \mathcal{L}(R)$ and $\mathcal{I}(R) \subseteq \text{FP-}\mathcal{I}(R) \subseteq \text{FP}_\infty\text{-}\mathcal{I}(R)$ imply that

$$① \leq ② \leq ③ \quad \text{and} \quad ④ \leq ⑤ \leq ⑥.$$

Suppose that $① < \infty$. We know from Bennis and Mahdou's result that $① = ④$. Moreover, $\mathrm{id}(\mathcal{P}(R)) < \infty$ and $\mathrm{pd}(\mathcal{I}(R)) < \infty$.

Assume first that M is strongly Gorenstein projective. By Remark 2.3 there is a short exact sequence $0 \to M \to P \to M \to 0$ with P is projective. The Horseshoe Lemma (see [10, Remark, page 187]) gives a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & P & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_0 & \to & I_0 \oplus I_0 & \to & I_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_n & \to & E_n & \to & I_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where I_i is injective for $i = 0, ..., n-1$. Since P is projective, $\mathrm{id}_R(P) \leq n$ (by Lemma 2.1); hence E_n is injective. On the other hand, from [7, Theorem 2.2], $\mathrm{pd}_R(E) \leq n$ for every injective R-module E. Then, $\mathrm{Ext}^i_R(E, I_n) = 0$ for all $i \geq n+1$. Then, from Remark 2.3, I_n is strongly Gorenstein injective, and so $\mathrm{Gid}_R(M) \leq n$. This implies, from [6, Proposition 2.19], that $\mathrm{Gid}_R(G) \leq n$ for any Gorenstein projective R-module G, since every Gorenstein projective R-module is a direct summand of a strongly Gorenstein projective R-module (Lemma 2.4).

Finally, consider an R-module M with $\mathrm{Gpd}_R(M) \leq m \leq n$. We can assume that $\mathrm{Gpd}_R(M) \neq 0$. Then, there exists a short exact sequence $0 \to K \to N \to M \to 0$ such that N is Gorenstein projective and $\mathrm{Gpd}_R(K) \leq m-1$ [6, Proposition 2.18]. By induction, $\mathrm{Gid}_R(K) \leq n$ and $\mathrm{Gid}_R(N) \leq n$. Therefore, using [6, Theorems 2.22 and 2.25] and the long exact sequence of Ext, we get that $\mathrm{Gid}_R(M) \leq n$. □

Notice also that $\mathrm{Gid}_{(\mathcal{Z},\mathcal{W})}(\mathcal{P}(R)) \leq \mathrm{id}(\mathcal{P}(R)) < \infty$ for any GI-admissible pair $(\mathcal{Z}, \mathcal{W}) \in \{(\mathcal{I}(R), \mathcal{I}(R)), (\text{FP-}\mathcal{I}(R), \mathcal{I}(R)), (\text{FP}_{\infty}\text{-}\mathcal{I}(R), \mathcal{I}(R))\}$.

Notice also that $\mathrm{Gid}_{(\mathcal{Z},\mathcal{W})}(\mathcal{P}(R)) \leq \mathrm{id}(\mathcal{P}(R)) < \infty$ for any GI-admissible pair $(\mathcal{Z},\mathcal{W}) \in \{(\mathcal{I}(R),\mathcal{I}(R)), (\text{FP-}\mathcal{I}(R),\mathcal{I}(R)), (\text{FP}_\infty\text{-}\mathcal{I}(R),\mathcal{I}(R))\}$.
So, from the previous theorem ① = ④ = ⑤ = ⑥.

By repeating the dual result with ④ $=$ ① $< \infty$, we get ④ $=$ ① $=$ ② $=$ ③.

By repeating the dual result with ④ $=$ ① $< \infty$, we get ④ $=$ ① $=$ ② $=$ ③.
Finally, notice that if some of ②, ③, ④, ⑤, ⑥ is finite then ① is finite.

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Thank you!