The deep locus of cluster varieties

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Plan

1. Cluster varieties and the deep locus
2. The deep locus and torus actions
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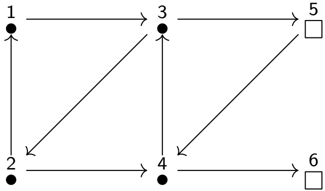
Cluster algebras and ($\mathcal{A}$-type) cluster varieties

Let Q be an ice quiver (i.e., some vertices are declared *frozen*). We assume that we have

- n mutable vertices.
- m frozen vertices (we may assume there are no arrows between them).

Then we can form the *extended exchange matrix* $\widetilde{B}$, and $(n+m)\times n$-matrix,

$$\widetilde{b}_{ij} := \#\{i \to j\} - \#\{j \to i\}.$$



$$\widetilde{B} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

A *seed* $t = (\mathbf{x}, \widetilde{B}) = (\mathbf{x}, Q)$ is the data of:

1. A set of algebraically independent variables $x_1, \ldots, x_n$, $x_{n+1}, \ldots, x_{n+m}$.
2. An ice quiver Q or, equivalently, an extended exchange matrix $\widetilde{B}$.

The *mutation* of a seed t in direction k $(1 \leq k \leq n)$ is the seed

$$\mu_k(t) = (\mu_k(\mathbf{x}), \mu_k(Q))$$

where

$$\mu_k(\mathbf{x}) = \mathbf{x} \setminus \{x_k\} \cup \{x_k'\}$$

and

$$x_k x_k' = \prod_{i \to k} x_i + \prod_{k \to j} x_j$$

And $\mu_k(Q)$ is obtained from Q via an explicit three-step procedure. We say that two seeds t and t' are mutation equivalent if one can be obtained from the other via iterated mutations.

Definition (Fomin-Zelevinsky)

The *cluster algebra* $A(Q) = A(\widetilde{B})$ is the $\mathbb{C}$-algebra $A(Q)$ generated by

$$\bigcup_{t \text{ is mutation equivalent to } (\mathbf{x}, Q)} \mathbf{x}^{(t')}, \qquad \text{and} \qquad x_{n+1}^{-1}, \ldots, x_{n+m}^{-1}.$$

Note that $A(Q) \subseteq \mathbb{C}(x_1, \ldots, x_{n+m})$.

Example

Take $Q = \bullet \to \square$. Then,

$$A(Q) = \mathbb{C}\left[x_1, \frac{1+x_1}{x_2}, x_2^{\pm 1}\right] = \mathbb{C}[x_1, x_1'][(x_1 x_1' - 1)^{-1}].$$

Theorem (The Laurent Phenomenon (Fomin-Zelevinsky))

Let $t = (\widetilde{\mathbf{x}}, \widetilde{Q})$ be a seed equivalent to $(\mathbf{x}, Q)$. Then,

$$A(Q) \subseteq \mathbb{C}[\widetilde{x}_1^{\pm 1}, \dots, \widetilde{x}_{n+m}^{\pm 1}],$$

or equivalently,

$$A(Q)[(\widetilde{x}_1 \cdots \widetilde{x}_{n+m})^{\pm 1}] = \mathbb{C}[\widetilde{x}_1^{\pm 1}, \dots, \widetilde{x}_{n+m}^{\pm 1}].$$

Geometrically, the nonvanishing locus of $\widetilde{x}_1, \dots, \widetilde{x}_{n+m}$ defines an open torus, called a *cluster torus*,

$$\mathbb{T}_t \subseteq \operatorname{Spec}(A(Q)).$$

Definition

- The ($\mathcal{A}$-type) cluster *variety* is

$$\mathcal{A}(Q) := \mathrm{Spec}\,(A(Q)).$$

- The cluster *manifold* is

$$\mathcal{M}(Q) := \bigcup \mathbb{T}_t \subseteq \mathcal{A}(Q).$$

- The deep locus is

$$\mathcal{D}(Q) := \mathcal{A}(Q) \setminus \mathcal{M}(Q).$$

Example

In the case $Q = \bullet \to \square$, $\mathcal{A}(Q) = \mathbb{C}^2 \setminus \{xy = 1\}$,
$\mathcal{M}(Q) = (\mathbb{C}^2 \setminus \{(0,0)\}) \setminus \{xy = 1\}$, $\mathcal{D}(Q) = \{(0,0)\}$.

The deep locus and torus actions

It is clear that

$$\mathrm{Sing}(\mathcal{A}(Q)) \subseteq \mathcal{D}(Q)$$

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To remedy this, let us consider a torus $\widetilde{T}$ of dimension $n + m$. The torus T acts on the initial seed $(x_1, \ldots, x_{n+m})$ by re-scaling the variables. This action does *not* extend in general to an action of T on the cluster algebra $A(Q)$. Consider however the morphism

$$\beta : \widetilde{T} \to (\mathbb{C}^\times)^n, \qquad (z_1, \ldots, z_{n+m}) \to \left(\prod_{i=1}^{n+m} z_i^{\widetilde{b}_{ij}} \right)_{j=1}^{n}.$$

Lemma (Gekhtman-Shapiro-Vainshtein)

The torus $T := \ker(\beta)$ *acts naturally on* $A(Q), \mathcal{A}(Q), \mathcal{M}(Q)$ *and* $\mathcal{D}(Q)$. *Moreover,* T *acts freely on* $\mathcal{M}(Q)$.

Thus, if $p \in \mathcal{A}(Q)$ and $Stab_T(p) \neq \{1\}$, necessarily $p \in \mathcal{D}(Q)$.

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Conjecture (V. Shende, D. Speyer)

$\mathcal{D}(Q) = \{p \in \mathcal{A}(Q) \mid Stab_T(p) \neq \{1\}\}$.

Lemma

Let $\widetilde{B}_1 = \begin{pmatrix} B \\ \hline C_1 \end{pmatrix}$ *and* $\widetilde{B}_2 = \begin{pmatrix} B \\ \hline C_2 \end{pmatrix}$ *be extended exchage matrices. Assume that*

$$\mathrm{Span}_{\mathbb{Z}}\{\textit{rows of } \widetilde{B}_1\} = \mathrm{Span}_{\mathbb{Z}}\{\textit{rows of } \widetilde{B}_2\}.$$

Then,

$$\mathcal{D}(\widetilde{B}_1) \times (\mathbb{C}^\times)^{m_2} \cong \mathcal{D}(\widetilde{B}_2) \times (\mathbb{C}^\times)^{m_1}.$$

Moreover, the Conjecture is true for $\widetilde{B}_1$ *if and only if it is true for* $\widetilde{B}_2$*.*

From now on, we will focus on the case

$$\mathrm{Span}_{\mathbb{Z}}\{\text{rows of } \widetilde{B}\} = \mathbb{Z}^n.$$

In this case, $T = (\mathbb{C}^\times)^m$.

Type ADE and beyond

Theorem (Fomin-Zelevinsky)

The cluster algebra $A(Q)$ has finite cluster type (i.e., there are only a finite number of seeds) if and only if Q^{uf} is mutation-equivalent to an orientation of a finite type Dynkin diagram.

Theorem (CGSS)

The conjecture is true for cluster algebras of finite cluster type (and $\mathbb{Z}$-full rank). Moreover, in this case we have:

- $\mathcal{D}(Q) = \emptyset$ *if* Q^{uf} *is of type* A_{2n}, E_6 *and* E_8.
- $\mathcal{D}(Q)$ *is nonempty, irreducible and smooth if* Q^{uf} *is of type* A_{2n+1}, D_{2n+1} *and* E_7.
- $\mathcal{D}(Q)$ *is nonempty, non-irreducible and not smooth if* Q^{uf} *is of type* D_{2n}, $2n \geq 4$. *It is not equidimensional if* $2n > 4$.

Remark

Recent results of Breyer-Muller describe the deep locus in finite cluster type ADE without frozen variables. Their methods are based on studying cluster algebras of (unpunctured) marked surfaces.

The idea of the proof is to use *explicit* geometric models of the cluster varieties via *(half decorated) double Bott-Samelson cells*. These algebraic varieties were introduced by Elek-Lu and Elek-Lu-Yu (2016, 2019). It follows from work of Goodearl-Yakimov (2018) that they admit cluster structures. The work of Shen-Weng (2019) constructs a very explicit cluster structure on them, and work of Casals-Gorsky-Gorsky-Le-Shen-S. (2022) constructs many cluster tori using weave calculus.

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Recall that a flag $F^\bullet$ on $\mathbb{C}^n$ is a sequence of subspaces $F^0 = \{0\} \subseteq F^1 \subseteq \cdots \subseteq F^n = \mathbb{C}^n$ with $\dim F^i = i$. Two flags $F^\bullet$ and $G^\bullet$ are said to be in position $i = 1, \ldots, n = 1$ if $G^i \neq F^i$ and $G^j = F^j$ for $j \neq i$. We denote this relation by $F^\bullet \xrightarrow{i} G^\bullet$. Finally, the *standard flag* $F^\bullet_{std}$ is given by

$$F^i_{std} = \langle e_1, \ldots, e_i \rangle \subseteq \mathbb{C}^n.$$

Definition

Let $\beta = \sigma_{i_1} \ldots \sigma_{i_\ell} \in \mathrm{Br}_n^+$ be a positive braid word on n strands. The *half-decorated double Bott-Samelson cell* $\mathrm{BS}(\beta)$ is

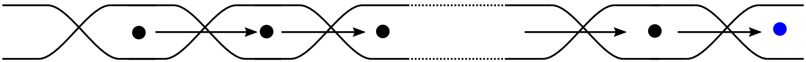
$$\mathrm{BS}(\beta) := \{(F_0^\bullet, \ldots, F_\ell^\bullet) \mid F_0^\bullet = F_{std}^\bullet, F_{j-1}^\bullet \xrightarrow{i_j} F_j^\bullet, F_\ell^\bullet \in B_- B_+ / B_+\}.$$

Remark

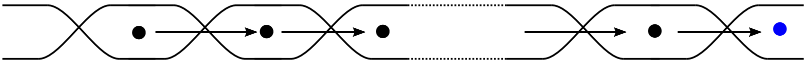
The double Bott-Samelson cells $\mathrm{BS}(\beta)$ *are special cases of braid varieties.*

Lemma

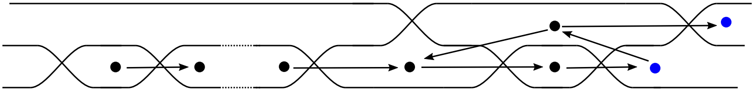
- *The cluster structure on* $\mathrm{BS}(\beta)$ *has* $\mathbb{Z}$*-full rank. (Shen-Weng).*
- *Note that the maximal torus in* PGL_n *acts diagonally on* $\mathrm{BS}(\beta)$*. This action coincides with the action by cluster automorphisms introduced above. (CGSS).*



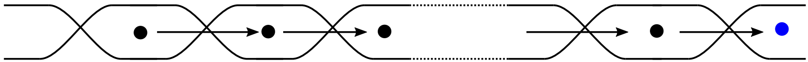
A_{n-1}



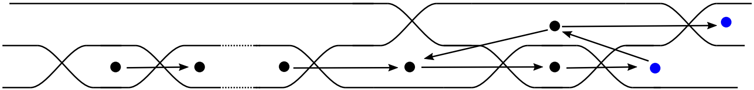
$$A_{n-1}$$



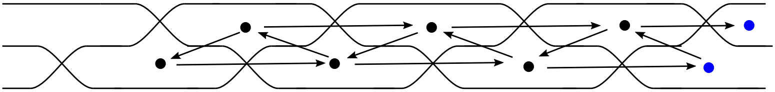
$$D_n$$



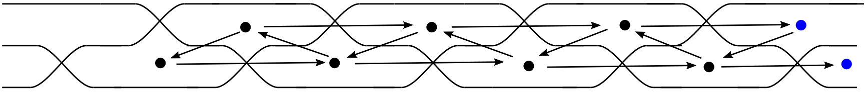
A_{n-1}



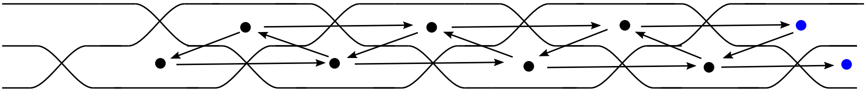
D_n



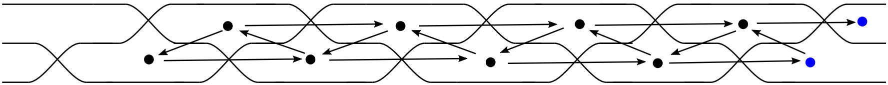
E_6



E_7



E_7



E_8

Conjecture

Let $\beta \in \mathrm{Br}_n^+$ be a positive braid, and T the maximal torus of PGL_n (note that we may need to take a smaller torus if not every Coxeter generator appears in β). The following are equivalent.

1. $\mathcal{D}(\mathrm{BS}(\beta)) = \emptyset$.
2. T acts freely on $\mathrm{BS}(\beta)$.
3. T acts freely on the frozen variables of $\mathrm{BS}(\beta)$.
4. The braid β closes up to a knot (i.e. with a single component).

We know that (2), (3) and (4) are equivalent, and that (2) $\Rightarrow$ (1). We know that (1) $\Rightarrow$ (2) for the braids above and for braid words of the form

$$\sigma_1^a(\sigma_2\sigma_1)^b.$$

This is already enough to show the following.

Proposition

Let $\Pi_{3,n}^\circ$ be the maximal positroid stratum in $\mathrm{Gr}(3, n)$. Then, $\mathcal{D}(\Pi_{3,n}^\circ) = \emptyset$ if and only if $\gcd(3; n) = 1$.

The previous proposition has an interpretation in terms of planar geometry. Up to a normalization (meaning that we can choose the fix the first three points):

$\Pi^{\circ}_{3,n} = \{(p_1, \ldots, p_n) \in (\mathbb{C}\mathrm{P}^2)^n \mid p_1p_2p_3, p_2p_3p_4, \ldots, p_np_1p_2$ are NOT collinear$\}$.

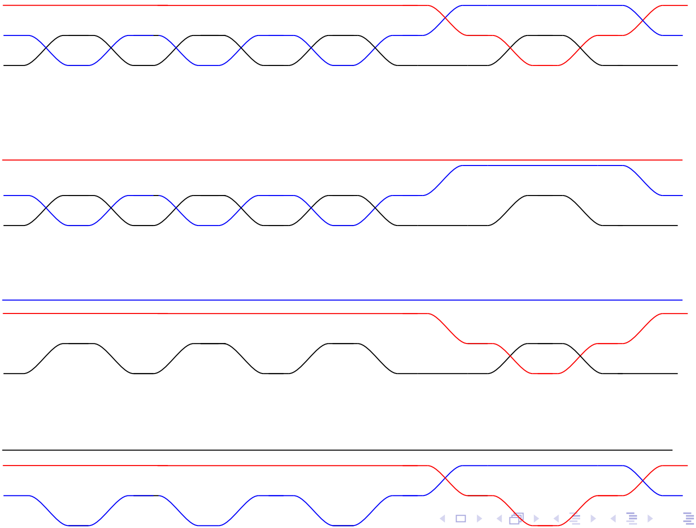
Then, an element $(p_1, \ldots, p_n) \in \Pi^{\circ}_{3,n}$ belongs to $\mathcal{D}(\Pi^{\circ}_{3,n})$ if and only if there exists a nontrivial decomposition

$$\mathbb{C}^3 = V_1 \oplus V_2$$

such that $p_i \subseteq V_1$ or $p_i \subseteq V_2$ for every $i = 1, \ldots, n$. Since either $\dim(V_1) = 1$ or $\dim(V_2) = 1$, this is only possible if 3 divides n.

Components

To find the components of $\mathcal{D}(\mathrm{BS}(\beta))$, we separate "one by one" the connected components of the link obtained by closing β. Let us exemplify this with type D_8.



Thanks for your attention!