# Weyl algebras for quantum homogeneous spaces and applications.

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Joint work with G. Letzter and S. Sahi

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# A little bit of history...

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \longrightarrow \mathrm{rdet(A)} = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

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$E_{i,j} = \sum_{a=1}^{n} x_{i,a} \partial_{j,a}$ act on polynomials in $n^2$ indeterminates $x_{i,j}$

***The* Capelli identity:** $\mathrm{rdet}\big[E_{j,i} + (n-i)\delta_{j,i}\big]_{i,j=1}^{n} = \det(X)\det(D).$

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# Background on Capelli operators

- $\mathfrak{g}$ : Reductive Lie algebra.
- $V$ : irreducible finite dimensional $\mathfrak{g}$-module.
- Assume that $\mathscr{P}(V)$ is multiplicity-free.

$$\mathscr{P}(V) \cong \bigoplus_{\lambda \in \Omega} V_\lambda$$

$\mathscr{PD}(V)$ is the "Weyl algebra" generated by the operators

$$f \mapsto x_i f \text{ and } f \mapsto \frac{\partial}{\partial x_i} f \quad 1 \leq i \leq \dim V$$

$\mathscr{PD}(V)^{\mathfrak{g}}$ : Algebra of $\mathfrak{g}$-invariant differential operators.

$$x \in \mathfrak{g}, D \in \mathscr{PD}(V) \longmapsto (x \cdot D)(f) = x \cdot (D(f)) - D(x \cdot f).$$

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# Spectrum of the Capelli basis

$$\mathscr{P}\mathscr{D}(V) \cong \mathscr{P}(V) \otimes \mathscr{D}(V)$$
$$\cong \bigoplus_{\lambda,\mu\in\Omega} V_\lambda \otimes V_\mu^* \cong \bigoplus_{\lambda,\mu\in\Omega} \mathrm{Hom}(V_\mu, V_\lambda)$$

$$\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda) := \begin{cases} \mathbb{C} & \text{if } \lambda = \mu, \\ \{0\} & \text{if } \lambda \neq \mu. \end{cases} \qquad D_\lambda \leftrightarrow 1 \in \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\lambda)$$

**Example**

- $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$, $V := \mathrm{Mat}_{n\times n}(\mathbb{C}) \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^*$.

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**Problem (Kostant, Sahi, Wallach,...):** Compute a formula for $d_\lambda(\mu)$.

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$$\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}} \cong \left(\mathscr{P}(V) \otimes \mathscr{D}(V)\right)^{\mathfrak{g}}$$
$$\cong \bigoplus_{\lambda,\mu\in\Omega} (V_\lambda \otimes V_\mu^*)^{\mathfrak{g}} \cong \bigoplus_{\lambda,\mu\in\Omega} \mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda)$$

$$\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda) := \begin{cases} \mathbb{C} & \text{if } \lambda = \mu, \\ \{0\} & \text{if } \lambda \neq \mu. \end{cases} \qquad D_\lambda \leftrightarrow 1 \in \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\lambda)$$

**Example**

- $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$, $V := \mathrm{Mat}_{n\times n}(\mathbb{C}) \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^*$.

$$\mathscr{P}(V) \cong \bigoplus_{\ell(\lambda)\leqslant n} V_\lambda \qquad V_\lambda \cong F_\lambda^* \otimes F_\lambda.$$

- $\lambda = (1) = (1, 0, \ldots, 0) \rightsquigarrow D_{(1)} = \sum_{1\leqslant i,j\leqslant n} x_{i,j} \frac{\partial}{\partial x_{i,j}}$
- $\lambda := (1^n) = (1, ..., 1) \rightsquigarrow D_{(1^n)} = \det([x_{i,j}]) \det([\frac{\partial}{\partial x_{i,j}}])$ Capelli operator

- $D_\lambda : \mathscr{P}(V) \to \mathscr{P}(V)$ is a $\mathfrak{g}$-module homomorphism ($D_\lambda$ is $\mathfrak{g}$-invariant).
- $\mathscr{P}(V)$ multiplicity-free $\Rightarrow$ $D_\lambda\big|_{V_\mu} = d_\lambda(\mu) I_{V_\mu}$ for all $\lambda, \mu$.

**Problem (Kostant, Sahi, Wallach,...):** Compute a formula for $d_\lambda(\mu)$.

# Interpolation Jack polynomials

- $P^\rho_\lambda(x_1,\ldots,x_n)$ : symmetric polynomial of degree $\leqslant |\lambda| := \lambda_1+\cdots+\lambda_n$.
- $\lambda = (\lambda_1,\lambda_2\ldots,\lambda_n)$, $\lambda_i \geqslant \lambda_{i+1}$, $\lambda_i \in \mathbb{Z}^{\geqslant 0}$.
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Let $\mathbb{F} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$. Set

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# Quantum symmetric spaces

- $\mathfrak{g}$: reductive Lie algebra , $\theta$: involution of $\mathfrak{g}$ $\rightsquigarrow$ $\mathfrak{k} := \mathfrak{g}^\theta$.
- $\mathcal{B}_\theta(b)$: deformation of $U(\mathfrak{g}^\theta)$ that fits in $U_q(\mathfrak{g})$ as a right coideal subalgebra.

**Generators of $\mathcal{B}_\theta(b)$**

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# Quantized coordinate ring of $\mathrm{Mat}_{N\times N}$

## The FRT bialgebra

- $R = \sum R_{ij}^{kl} E_{k,i} \otimes E_{l,j} \in \mathrm{End}_{\Bbbk}(\Bbbk^N \otimes \Bbbk^N)$.
- $\mathcal{A}(R) := \langle t_{ij}, 1 \leqslant i,j \leqslant N \ : \ RT_1T_2 = T_2T_1R \rangle$

$$T := [t_{ij}] \quad , \quad T_1 = T \otimes I \quad , \quad T_2 = I \otimes T$$

## The algebra $\mathscr{P}_{N\times N}$

- Set $\mathscr{P}_{N\times N} := \mathcal{A}(R)$ for the $R$-matrix given by:

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# The bimodule algebras $\mathscr{P}_{N\times N}$ and $\mathscr{D}_{N\times N}$

- $\mathscr{P}_{N\times N} \hookrightarrow U_q(\mathfrak{gl}_N)^\circ$ (matrix coefficients of the "standard module")
- $\mathscr{P}_{N\times N}$ is a bialgebra:

$$\Delta(t_{i,j}) = \sum_k t_{i,k} \otimes t_{k,j}.$$

- $U_q(\mathfrak{gl}_N)$ acts on $\mathscr{P}_{N\times N}$ by left and right translation:

$$x \cdot_L u := \sum \langle u_1, x\rangle u_2 \quad , \quad x \cdot_R u = \sum u_1 \langle u_2, x^\natural\rangle,$$

  where $x \mapsto x^\natural$ is the Hopf isomorphism $U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N)^{\mathrm{op}}$ given by

$$E_i^\natural := q^{-1} F_i K_i \ , \ F_i^\natural := q K_i^{-1} E_i \ , \ K_{\varepsilon_i}^\natural := K_{\varepsilon_i},$$

  and $\Delta(u) = \sum u_1 \otimes u_2$ (Sweedler notation).
- Analogous properties hold by $\mathscr{D}_{N\times N} := (\mathscr{P}_{N\times N})^{\mathrm{op}}$.

**Formulas for left translation of $U_q(\mathfrak{gl}_N)$ on $\mathscr{P}_{N\times N}$ and $\mathscr{D}_{N\times N}$**

$$E_k \cdot t_{i,j} = \delta_{k,i-1} t_{i-1,j} \quad , \quad F_k \cdot t_{i,j} = \delta_{k,i} t_{i+1,k} \quad , \quad K_{\varepsilon_k} \cdot t_{i,j} = q^{\delta_{k,j}} t_{i,j}$$

$$E_k \cdot \partial_{i,j} = -q^{-1} \partial_{i+1,j} \quad , \quad F_k \cdot \partial_{i,j} = -q \delta_{i-1,k} \partial_{i-1,j} \quad , \quad K_{\varepsilon_k} \cdot t_{i,j} = q^{-\delta_{k,j}} \partial_{i,j}$$

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- $\mathscr{P}_{N\times N}$ is a bialgebra:
$$\Delta(t_{i,j}) = \sum_k t_{i,k} \otimes t_{k,j}.$$
- $U_q(\mathfrak{gl}_N)$ acts on $\mathscr{P}_{N\times N}$ by left and right translation:
$$x \cdot_L u := \sum \langle u_1, x\rangle u_2 \quad , \quad x \cdot_R u = \sum u_1 \langle u_2, x^\natural\rangle,$$
where $x \mapsto x^\natural$ is the Hopf isomorphism $U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N)^{\mathrm{op}}$ given by
$$E_i^\natural := q^{-1}F_iK_i \ , \ F_i^\natural := qK_i^{-1}E_i \ , \ K_{\varepsilon_i}^\natural := K_{\varepsilon_i},$$
and $\Delta(u) = \sum u_1 \otimes u_2$ (Sweedler notation).
- Analogous properties hold by $\mathscr{D}_{N\times N} := (\mathscr{P}_{N\times N})^{\mathrm{op}}$.

**Formulas for left translation of $U_q(\mathfrak{gl}_N)$ on $\mathscr{P}_{N\times N}$ and $\mathscr{D}_{N\times N}$**

$$E_k \cdot t_{i,j} = \delta_{k,i-1}t_{i-1,j} \quad , \quad F_k \cdot t_{i,j} = \delta_{k,i}t_{i+1,k} \quad , \quad K_{\varepsilon_k} \cdot t_{i,j} = q^{\delta_{k,j}}t_{i,j}$$

$$E_k \cdot \partial_{i,j} = -q^{-1}\partial_{i+1,j} \quad , \quad F_k \cdot \partial_{i,j} = -q\delta_{i-1,k}\partial_{i-1,j} \quad , \quad K_{\varepsilon_k} \cdot t_{i,j} = q^{-\delta_{k,j}}\partial_{i,j}$$

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# Quantum homogeneous coordinate rings

Recall that $\mathfrak{k} = \mathfrak{g}^\theta$. Set

$$\mathscr{P} := \begin{cases} \mathscr{P}_{N\times N} & \mathbb{F} \neq \mathbb{C}, \\ \mathscr{P}_{n\times n} \otimes \mathscr{P}_{n\times n} & \mathbb{F} = \mathbb{C}. \end{cases}, \quad \mathscr{D} := \begin{cases} \mathscr{D}_{N\times N} & \mathbb{F} \neq \mathbb{C}, \\ \mathscr{D}_{n\times n} \otimes \mathscr{D}_{n\times n} & \mathbb{F} = \mathbb{C}. \end{cases}$$

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**Proposition**

The $x_{i,j}$ and the $d_{i,j}$ are right $\mathcal{B}_\theta(b)$-invariant if and only if

- $\mathbb{F} = \mathbb{R}$: $b_i = 1$ for all $1 \leqslant i \leqslant n$.
- $\mathbb{F} = \mathbb{C}$: $b_i = q$ for all $1 \leqslant i \leqslant n$.
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**Theorem (LSS)**

$\mathscr{P}_\theta$ is the algebra generated by the $x_{i,j}$ modulo the following relations:

- $R_{\mathfrak{g}} X_1 R_{\mathfrak{g}}^{t_1} X_2 = X_2 R_{\mathfrak{g}}^{t_1} X_1 R_{\mathfrak{g}}$ where $X_1 = X \otimes I$ , $X_2 = I \otimes X$.
- $x_{i,j} = \gamma x_{j,i}$ for $i < j$ and $x_{a,b} = 0$ for $(a,b) \in \mathcal{S}$ where

$$\gamma := \begin{cases} q & \mathbb{F} = \mathbb{R} \\ 1 & \mathbb{F} = \mathbb{C} \\ -q^{-1} & \mathbb{F} = \mathbb{H} \end{cases} \quad , \quad \mathcal{S} = \begin{cases} \varnothing & \mathbb{F} = \mathbb{R} \\ \{(i,j), 1 \leqslant i,j \leqslant n\} \cup \{(i,j), n+1 \leqslant i,j \leqslant 2n\} & \mathbb{F} = \mathbb{C} \\ \{(i,i), i = 1, \ldots, 2n\} & \mathbb{F} = \mathbb{H} \end{cases}$$

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**Theorem (LSS)**

$\mathscr{D}_\theta$ is the algebra generated by the $\partial_{i,j}$ modulo the following relations:

- $R_\mathfrak{g} D_2 R_\mathfrak{g}^{t_1} D_1 = D_1 R_\mathfrak{g}^{t_1} D_2 R_\mathfrak{g}$ where $D_1 = D \otimes I$ , $D_2 = I \otimes D$.
- $\partial_{i,j} = \gamma \partial_{j,i}$ for $i < j$ and $\partial_{a,b} = 0$ for $(a,b) \in \mathcal{S}$ where

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# Towards the quantum Weyl algebra

**Twisted tensor product**

Let $A$ and $B$ be associative algebras and let $\tau : B \otimes A \to A \otimes B$ satisfy

- $\tau(1 \otimes a) = a \otimes 1$ and $\tau(b \otimes 1) = 1 \otimes b$.
- $\tau \circ (1_B \otimes m_A) = (m_A \otimes 1_B)(1_A \otimes \tau)(\tau \otimes 1_A)$.
- $\tau \circ m_B \otimes 1_A = (1_A \otimes m_B)(\tau \otimes 1_B)(1_B \otimes \tau)$.

Then $A \otimes B$ is an associative algebra when equipped with

$$m_{A \otimes B} := (m_A \otimes m_B) \circ (1_A \otimes \tau \otimes 1_B).$$

**Pairing**

A pairing between bialgebras $A$ and $B$ is a bilinear map from $A \times B$ to scalars such that

- $\langle a, 1 \rangle = \epsilon(a)$ , $\langle 1, b \rangle = \epsilon(b)$.
- $\langle \Delta(a), b \otimes b' \rangle = \langle a, bb' \rangle$ , $\langle a \otimes a', \Delta(b) \rangle = \langle aa', b \rangle$.

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- $\tau \circ (1_B \otimes m_A) = (m_A \otimes 1_B)(1_A \otimes \tau)(\tau \otimes 1_A)$.
- $\tau \circ m_B \otimes 1_A = (1_A \otimes m_B)(\tau \otimes 1_B)(1_B \otimes \tau)$.

Then $A \otimes B$ is an associative algebra when equipped with

$$m_{A \otimes B} := (m_A \otimes m_B) \circ (1_A \otimes \tau \otimes 1_B).$$

## Pairing

A pairing between bialgebras $A$ and $B$ is a bilinear map from $A \times B$ to scalars such that

- $\langle a, 1 \rangle = \epsilon(a)$ , $\langle 1, b \rangle = \epsilon(b)$.
- $\langle \Delta(a), b \otimes b' \rangle = \langle a, bb' \rangle$ , $\langle a \otimes a', \Delta(b) \rangle = \langle aa', b \rangle$.

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# Towards the quantum Weyl algebra

**Lemma**

Let $A$ and $B$ be bialgebras. Suppose that

- $\mathbf{u}\langle\cdot,\cdot\rangle$ is a pairing between $A^{\mathrm{op}}$ and $B$.
- $\mathbf{v}\langle\cdot,\cdot\rangle$ is a pairing between $A$ and $B^{\mathrm{op}}$.

Then the map $\tau(b\otimes a) := \sum a_2 \otimes b_2 \mathbf{v}\langle a_1, b_1\rangle \mathbf{u}\langle a_3, b_3\rangle$ is a twisting map.

**Two pairs of pairings**

Recall the $R$-matrix defined by $R^{ii}_{ii} = q$, $R^{ij}_{ij} = 1$ for $i \neq j$, $R^{ji}_{ij} = (q - q^{-1})$ for $i < j$.

- Set $R_0 := R^{t_2}$ an $R_1 := (R_{21}^{-1})^{t_2}$.
- For $\sigma \in \{0,1\}$ we have pairings $\mathbf{u}_\sigma\langle\cdot,\cdot\rangle$ between $\mathscr{D}_{N\times N}$ and $\mathscr{P}_{N\times N}$, and $\mathbf{v}_\sigma\langle\cdot,\cdot\rangle$ between $\mathscr{P}_{N\times N}$ and $\mathscr{D}_{N\times N}$, defined by

$$\mathbf{u}_\sigma(\partial_{i,j}, t_{k,l}\rangle = \mathbf{v}_\sigma\langle t_{i,j}, \partial_{k,l}\rangle = (R_\sigma)^{ik}_{jl}.$$

**Proposition**

Let $\tau := \tau_{\sigma,\sigma'}$ for $\sigma, \sigma' \in \{0,1\}$. The twisted tensor product $A_{\sigma,\sigma'} := \mathscr{P}_{N\times N} \otimes_\tau \mathscr{D}_{N\times N}$ is a $U_q(\mathfrak{gl}_N)$-bimodule algebra.

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**Remark**

We have $\mathcal{A}_{0,0} \cong \mathcal{A}_{1,1}$ and $\mathcal{A}_{0,1} \cong \mathcal{A}_{1,0}$ via the maps $t_{i,j} \leftrightarrow t_{N+1-i,N+1-j}$ and $\partial_{i,j} \leftrightarrow \partial_{N+1-i,N+1-j}$.

- $\hat{\mathscr{P}}$: algebra generated by two copies of $\mathscr{P}$, no relations ($t_{i,j}$ and $t'_{i,j}$).
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**Extended Weyl algebras**

For $\alpha, \beta, \upsilon, \sigma \in \{0, 1\}$ we set

- $\tau_{\alpha,\beta,\upsilon,\sigma}(\partial_{e,a} \otimes t_{f,b}) := \tau_{\alpha,\beta}(\partial_{e,a} \otimes t_{f,b})$
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$$\mathcal{A}_{\alpha,\beta,\upsilon,\sigma} := \hat{\mathscr{D}} \otimes_\tau \mathscr{P}$$

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# Towards the quantum Weyl algebra

**Proposition**

Set $d'_{i,j} := \sum q^{-2\hat{s}} J_{r,s} \partial_{i,r} \partial'_{j,s}$. Also set $S_\gamma = R_{\mathfrak{g}}$ if $\gamma = 0$ and $S_\gamma = (R_{\mathfrak{g}})^{-1}_{21}$ if $\gamma = 1$. If $\beta + \sigma = 1$, then the subalgebra of $\mathcal{A}_{\alpha,\beta,\upsilon,\sigma}$ generated by $\mathscr{P}'_\theta$ and $\mathscr{D}_\theta$ satisfies the relations

$$d_{ab} x'_{ef} = \sum_{r,w,p,q,x,y,m,l} (S_\alpha^{t_2})^{wr}_{xq} (S_\alpha^{t_2})^{pq}_{ma} (S_\upsilon^{t_2})^{xy}_{fl} (S_\upsilon^{t_2})^{ml}_{eb} x'_{pw} d_{ry}$$

for all $a, b, e, f$.

**Theorem (LSS)**

The relation

$$\tau^\theta_{\alpha,\upsilon}(d_{ab} \otimes x_{ef}) = \sum_{r,w,p,q,x,y,m,l} (S_\alpha^{t_2})^{wr}_{xq} (S_\alpha^{t_2})^{pq}_{ma} (S_\upsilon^{t_2})^{xy}_{fl} (S_\upsilon^{t_2})^{ml}_{eb} (x_{pw} \otimes d_{ry})$$

uniquely defines a twisting on $\mathscr{D}_\theta \otimes \mathscr{P}_\theta$. The twisted tensor product $\mathcal{A}^\theta_{\upsilon,\sigma} := \mathscr{P}_\theta \otimes_{\tau^\theta_{\upsilon,\sigma}} \mathscr{D}_\theta$ naturally embeds in $\mathcal{A}_{\alpha,\beta,\upsilon,\sigma}$ for $\beta + \sigma = 1$.

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# PBW deformation of $\mathcal{A}_{\upsilon,\upsilon}$

- $A = T(Y)/\langle I \rangle$, $B = T(Z)/\langle J \rangle$ Koszul algebras, $I \subseteq Y^{\otimes 2}$, $J \subseteq Z^{\otimes 2}$.
- $\tau_{(1,1)} : Z \otimes Y \to Y \otimes Z$
- $\tau_{(m,m')} : Z^{\otimes m} \otimes Y^{\otimes m'} \to Y^{\otimes m'} \otimes Z^{\otimes m}$
- The induced $\tau : T(Z) \otimes T(Y) \to T(Y) \otimes T(Z)$ is a twisting map.

**Proposition (Walton–Witherspoon, 2018)**

Assume that $\tau$ descends to a twisting map $B \otimes A \to A \otimes B$. Then:

- $E := A \otimes_\tau B$ is Koszul.
- $E \cong T(Z \oplus Y)/\langle I + J + K \rangle$ where

$$K := \{w - \tau(w) \, : \, w \in Z \otimes Y\} \subseteq Z \otimes Y + Y \otimes Z.$$

**Theorem (Walton–Witherspoon, 2018)**

Let $\mu \in (I + J + K)^*$ and set $E_\mu := T(Z \oplus Y)/\langle r - \mu(r) \, : \, r \in I + J + K \rangle$. Then $E_\mu$ is a PBW deformation of $E$ if and only if $\mu \otimes 1 = 1 \otimes \mu$ on

$$(I + J + K) \otimes (Z + Y) \cap (Z + Y) \otimes (I + J + K).$$

# PBW deformation of $\mathcal{A}_{v,v}$

- $A = T(Y)/\langle I\rangle$, $B = T(Z)/\langle J\rangle$ Koszul algebras, $I \subseteq Y^{\otimes 2}$, $J \subseteq Z^{\otimes 2}$.
- $\tau_{(1,1)} : Z \otimes Y \to Y \otimes Z$
- $\tau_{(m,m')} : Z^{\otimes m} \otimes Y^{\otimes m'} \to Y^{\otimes m'} \otimes Z^{\otimes m}$
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# PBW deformation of $\mathcal{A}_{\upsilon,\upsilon}$

**The quantum Weyl algebra $\mathscr{P}\mathscr{D}_{N\times N}$ (Shklyarov, Sinelshchikov, Vaksman, 1999)**

- $Y$: linear span of the $t_{i,j}$, i.e., $A \cong \mathscr{P}_{N\times N}$.
- $Z$: linear span of the $\partial_{i,j}$, i.e., $B \cong \mathscr{D}_{N\times N}$.
- $\mu \equiv 0$ on $Y\otimes Y + Z\otimes Z + Y\otimes Z$, $\mu(\partial_{i,j}\otimes t_{k,l}) = \delta_{i,k}\delta_{j,s}$.
- $\tau := \tau_{\upsilon,\upsilon}$.

The resulting algebra is $\mathscr{P}\mathscr{D}_{N\times N}$ generated by the $t_{i,j}$ and the $\partial_{i,j}$.
The "relations in $K$" are generated by:

(i) $\partial_{cb}t_{da} = t_{da}\partial_{cb}$ if $b\neq a$ and $c\neq d$.

(ii) $\partial_{cb}t_{ca} = qt_{ca}\partial_{cb} + \sum_{c'>c}(q-q^{-1})t_{c'a}\partial_{c'b}$ if $b\neq a$.

(iii) $\partial_{ca}t_{da} = qt_{da}\partial_{ca} + \sum_{a'>a}(q-q^{-1})t_{da'}\partial_{ca'}$ if $c\neq d$.

(iv) $\partial_{ca}t_{ca} = 1 + \sum_{c'\geqslant c}\sum_{a'\geqslant a} q^{\delta_{c',c}+\delta_{a',a}}(q-q^{-1})^{2-\delta_{c',c}-\delta_{a',a}}t_{c',a'}\partial_{c',a'}$.

# PBW deformation of $\mathcal{A}_{\upsilon,\upsilon}$

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- $\mu \equiv 0$ on $Y \otimes Y + Z \otimes Z + Y \otimes Z$, $\mu(\partial_{i,j} \otimes t_{k,l}) = \delta_{i,k}\delta_{j,s}$.
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**Theorem (LSS)**

Let $\upsilon \in \{0,1\}$. Then the following hold:

- There exists an essentially unique left $U_q(\mathfrak{g})$ and right $\mathcal{B}_\theta$ invariant form on $\mathscr{D}_\theta \times \mathscr{P}_\theta$ such that

$$\langle d_{i,j}, x_{k,l}\rangle = q^{-\delta_{k,l}}\delta_{i,k}\delta_{j,l},$$

  under the constraints

$$\begin{cases} i \leqslant j \text{ and } k \leqslant l & \mathbb{F}=\mathbb{R} \\ i \leqslant n \text{ and } j \geqslant n+1 & \mathbb{F}=\mathbb{C} \\ i < j \text{ and } j < l & \mathbb{F}=\mathbb{H}. \end{cases}$$

- For $Y := \operatorname{span}\{x_{i,j}\}$, $Z := \operatorname{span}\{d_{i,j}\}$, $\mu \in (Z \otimes Y)^*$ defined using $\langle\cdot,\cdot\rangle$ and $\tau := \tau^{\theta}_{\upsilon,\upsilon}$ we obtain a PBW deformation $\mathscr{P}\mathscr{D}_\theta := E_\mu$ of $\mathscr{P}_\theta \otimes_\tau \mathscr{D}_\theta$ with a $U_q(\mathfrak{g})$-equivariant action

$$\mathscr{P}\mathscr{D}_\theta \otimes \mathscr{P}_\theta \to \mathscr{P}_\theta.$$

# PBW deformation of $\mathcal{A}^\theta_{\upsilon,\upsilon}$

**Theorem (LSS)**

Let $\upsilon \in \{0,1\}$. Then the following hold:

- There exists an essentially unique left $U_q(\mathfrak{g})$ and right $\mathcal{B}_\theta$ invariant form on $\mathscr{D}_\theta \times \mathscr{P}_\theta$ such that

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- For $Y := \operatorname{span}\{x_{i,j}\}$, $Z := \operatorname{span}\{d_{i,j}\}$, $\mu \in (Z \otimes Y)^*$ defined using $\langle \cdot, \cdot \rangle$ and $\tau := \tau^\theta_{\upsilon,\upsilon}$ we obtain a PBW deformation $\mathscr{P}\mathscr{D}_\theta := E_\mu$ of $\mathscr{P}_\theta \otimes_\tau \mathscr{D}_\theta$ with a $U_q(\mathfrak{g})$-equivariant action

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# Capelli operators: $q$-variation

- $\mathscr{P}_\theta \cong \bigoplus_{\ell(\lambda)\leqslant n} V_\lambda$, $\mathscr{D}_\theta \cong \bigoplus_{\ell(\lambda)\leqslant n} V_\lambda^*$ as $U_q(\mathfrak{g})$-modules.
- $D_\lambda \in (V_\lambda \otimes V_\lambda^*)^{U_q(\mathfrak{g})} \subseteq \mathscr{P}\mathscr{D}_\theta$.

**Interpolation Macdonald polynomials (Sahi, 1996)**

Let $\lambda$ be a partition satisfying $\ell(\lambda) \leqslant n$. Then $\mathsf{R}_\lambda \in \mathbb{Q}(\mathsf{q},\mathsf{t})[\mathsf{x}_1,\ldots,\mathsf{x}_n]$ is the unique polynomial satisfying the following:

- $\deg(\mathsf{R}_\lambda) = |\lambda|$.
- $\mathsf{R}_\lambda$ is symmetric in the $x_i$. Furthermore $\mathsf{R}_\lambda = m_\lambda + \sum_{\mu\leqslant\lambda} c_{\lambda,\mu} m_\mu$.
- $\mathsf{R}_\lambda(\mathsf{q}^{-\mu}\tau,\mathsf{q},\mathsf{t}) = 0$, where $\tau = (\tau_1,\ldots,\tau_n)$ with $\tau_i := \mathsf{t}^{-n+i}$, for $\mu$ such that $|\mu| \leqslant |\lambda|$ and $\mu \neq \lambda$.
- **Theorem.** The top degree homogeneous component of $\mathsf{R}_\lambda$ is the (usual) Macdonald polynomial $\mathsf{P}_\lambda$.

**Theorem (LSS)**

The eigenvalue of $D_\lambda \in \mathscr{P}\mathscr{D}_\theta$ on the irreducible component $F_\mu$ of $\mathscr{P}_\theta$ is equal to $\mathsf{R}_\lambda(\mathsf{q}^{-\mu}\tau,\mathsf{q},\mathsf{q}^{2/d})$ where $d := \dim \mathbb{F}$.

# Capelli operators: $q$-variation

- $\mathscr{P}_\theta \cong \bigoplus_{\ell(\lambda)\leqslant n} V_\lambda$, $\mathscr{D}_\theta \cong \bigoplus_{\ell(\lambda)\leqslant n} V_\lambda^*$ as $U_q(\mathfrak{g})$-modules.
- $D_\lambda \in (V_\lambda \otimes V_\lambda^*)^{U_q(\mathfrak{g})} \subseteq \mathscr{P}\mathscr{D}_\theta$.

**Interpolation Macdonald polynomials (Sahi, 1996)**

Let $\lambda$ be a partition satisfying $\ell(\lambda) \leqslant n$. Then $\mathsf{R}_\lambda \in \mathbb{Q}(\mathsf{q},\mathsf{t})[\mathsf{x}_1,\ldots,\mathsf{x}_n]$ is the unique polynomial satisfying the following:

- $\deg(\mathsf{R}_\lambda) = |\lambda|$.
- $\mathsf{R}_\lambda$ is symmetric in the $x_i$. Furthermore $\mathsf{R}_\lambda = m_\lambda + \sum_{\mu\leqslant\lambda} c_{\lambda,\mu} m_\mu$.
- $\mathsf{R}_\lambda(\mathsf{q}^{-\mu}\tau,\mathsf{q},\mathsf{t}) = 0$, where $\tau = (\tau_1,\ldots,\tau_n)$ with $\tau_i := \mathsf{t}^{-n+i}$, for $\mu$ such that $|\mu| \leqslant |\lambda|$ and $\mu \neq \lambda$.
- **Theorem.** The top degree homogeneous component of $\mathsf{R}_\lambda$ is the (usual) Macdonald polynomial $\mathsf{P}_\lambda$.

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# Capelli operators: $q$-variation

**The locally finite part of $U_q(\mathfrak{gl}_N)$, à la Joseph et Letzter (1994)**

- $F(\mathfrak{gl}_N) := \{x \in U_q(\mathfrak{gl}_N) \ : \ \dim_{U_q(\mathfrak{gl}_N)}(x) < \infty\}$.
- $F(\mathfrak{gl}_N) = \bigoplus_{\lambda \in \Omega_N} \mathrm{ad}_{U_q(\mathfrak{sl}_N)}(K_{2\lambda})$ where

$$\Omega_N = \left\{ \sum_i \lambda_i \varepsilon_i \ : \ \ell_{i+1} - \ell_i \in \{0, 2, 4, \ldots\} \right\}.$$

Furthermore, each summand $\mathrm{ad}_{U_q(\mathfrak{sl}_N)}(K_{2\lambda})$ contains a unique (up to scalar) $z_\lambda \in Z(U_q(\mathfrak{gl}_N))$.

**Proposition (LSS)**

There exists a subalgebra $F'(\mathfrak{gl}_N) \subseteq F(\mathfrak{gl}_N)$ together with an injection $F'(\mathfrak{gl}_N) \hookrightarrow \mathscr{P}\mathscr{D}_{N \times N}$ resulting in a commutative diagram

$$\begin{array}{ccc} F'(\mathfrak{gl}_N) \otimes \mathscr{P}_{N \times N} & \longrightarrow & \mathscr{P}\mathscr{D}_{N \times N} \otimes \mathscr{P}_{N \times N} \\ & \searrow & \downarrow \\ & & \mathscr{P}_{N \times N} \end{array}$$

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# The First Fundamental Theorem of Invariant Theory

- $G := \mathrm{GL}_n(\mathbb{C})$ , $V := \mathbb{C}^n$ , $G$ naturally acts on $V$ (hence on $V^*$).
- $\mathcal{E}_{k,l} := V^{\oplus k} \oplus (V^*)^{\oplus l}$.
- $G$ acts on the polynomial algebra $\mathcal{P}\left(\mathcal{E}_{k,l}\right)$:

$$g \cdot \phi(x) := \phi(g^{-1} \cdot x) \quad \text{for } g \in G,\ x \in \mathcal{E}_{k,l}.$$

- **Classical Problem:** Find concrete generators and relations for $\mathcal{P}(\mathcal{E}_{k,l})^G$.
- Define $\phi_{i,j}(v_1, \ldots, v_k, v_1^*, \ldots, v_l^*) := \langle v_j^*, v_i \rangle$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$.

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**Theorem (Schur, Weyl, ...)**

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# The operator commutant FFT (à la Roger Howe)

- $\mathcal{P} := \mathcal{P}_{m\times n} := \mathcal{P}(\mathrm{Mat}_{m\times n})$.
- $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ naturally acts on $\mathrm{Mat}_{m\times n}$:

$$(g, g') \cdot X := (g^{-1})^T X g'$$

- Passing to Lie algebras, we obtain an action of $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ on $\mathrm{Mat}_{m\times n}$:

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- $\mathrm{GL}_m \times \mathrm{GL}_n$ and $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ naturally act on $\mathcal{P}$.
- **Observation:** $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ act by *polarization* operators:

$$E_{ij} \rightsquigarrow \sum_{r=1}^{n} x_{ir} \frac{\partial}{\partial x_{jr}} \quad \text{for } 1 \leqslant i, j \leqslant m$$

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**Weyl algebra on $\mathrm{Mat}_{m\times n}$**

- $\mathcal{PD} := \mathcal{PD}_{m\times n}$: the $\mathbb{C}$-algebra generated by the $x_{i,j}$ and the $\partial_{i,j}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, modulo the relations

$$\partial_{i,j}x_{k,l} - x_{k,l}\partial_{i,j} = \delta_{i,k}\delta_{j,l}.$$

- $\mathcal{P}$ is a $\mathcal{PD}$-module.

- Set $U(\mathfrak{g}) := T(\mathfrak{g})/I$, where $I$ is the ideal generated by the elements

$$x \otimes y - y \otimes x - [x, y] \quad , \quad x, y \in \mathfrak{g}.$$

*Every Lie theorist knows:* $\quad \mathfrak{g} - \mathrm{mod} \cong U(\mathfrak{g}) - \mathrm{mod}.$

- Set $U_{m,n} := U(\mathfrak{gl}_m) \otimes U(\mathfrak{gl}_n)$. The polarization operators result in an algebra homomorphism

$$\phi_{m,n} : U_{m,n} \to \mathcal{PD},$$

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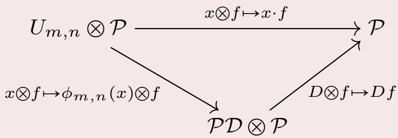

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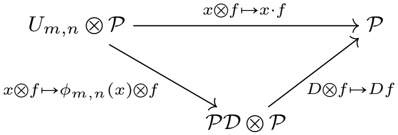


**Theorem (Howe, 1995)**

(i) The subalgebras $\phi_{m,n}(U(\mathfrak{gl}_m) \otimes 1)$ and $\phi_{m,n}(1 \otimes U(\mathfrak{gl}_n))$ of $\mathcal{PD}$ are mutual centralizers in $\mathcal{PD}$.

(ii) The FFT for $\mathrm{GL}_n(\mathbb{C})$ follows readily from (i).

**Remark (trivial)**

The mutual centralizer property does not hold in $\mathrm{End}(\mathcal{P})$.

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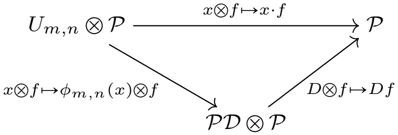


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# Double centralizer property: the $q$-version

$$U_L := U_q(\mathfrak{gl}_m)\ ,\ U_R := U_q(\mathfrak{gl}_n)\ ,\ U_{LR} := U_L \otimes U_R.$$

$$U_{LR} \xrightarrow{\ \phi_U\ } \mathrm{End}(\mathscr{P}_{m\times n}) \xleftarrow{\ \phi_{PD}\ } \mathscr{P}\mathscr{D}_{m\times n}$$

Set

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**Theorem (LSS)**

(i) $\mathcal{L}$ and $\mathcal{R}$ are mutual centralizers in $\mathscr{P}\mathscr{D}_{m\times n}$.

(ii) $\mathcal{L}$ is generated by the $\mathsf{L}_{i,j}$, $\leqslant i,j \leqslant m$.

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# Double centralizer property: the $q$-version

$$U_L := U_q(\mathfrak{gl}_m)\ ,\ U_R := U_q(\mathfrak{gl}_n)\ ,\ U_{LR} := U_L \otimes U_R.$$

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# FFT : the $q$-version

## Comparing with the FFT by Lehrer–Zhang–Zhang (2010)

Algebra proposed by L–Z–Z:

- Generators: $t_{ij}$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$; $\partial_{i,j}$ for $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant n$.
- Relations between the $t_{ij}$ (or between the $\partial_{ij}$): same as those of $\mathrm{gr}(\mathscr{A}_{k,l,n})$.
- Mixed relations:

  (i) $\partial_{cb} t_{da} = t_{da} \partial_{cb} \quad \text{if } b \neq a.$

  (ii) $\partial_{ca} t_{ba} = q t_{ba} \partial_{ca} + (q - q^{-1}) \sum_{a' > a} t_{ba'} \partial_{ca'}$

## Mixed relations of $\mathrm{gr}(\mathscr{A}_{k,l,n})$

(i) $\partial_{cb} t_{da} = t_{da} \partial_{cb} \quad \text{if } b \neq a \text{ and } c \neq d.$

(ii) $\partial_{cb} t_{ca} = q t_{ca} \partial_{cb} + \sum_{c' > c} (q - q^{-1}) t_{c'a} \partial_{c'b} \quad \text{if } b \neq a.$

(iii) $\partial_{ca} t_{da} = q t_{da} \partial_{ca} + \sum_{a' > a} (q - q^{-1}) t_{da'} \partial_{ca'} \quad \text{if } c \neq d.$

(iv) $$\partial_{ca} t_{ca} = \sum_{c' \geqslant c} \sum_{a' \geqslant a} q^{\delta_{c',c} + \delta_{a',a}} (q - q^{-1})^{2 - \delta_{c',c} - \delta_{a',a}} t_{c',a'} \partial_{c',a'}.$$