Categorical Connections

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26 August 2023

A von Neumann algebra A is a factor if $A \cap A' = \mathbb{C} \cdot Id$.
Suppose $A \subset B$ is a subfactor, ie a unital inclusion of type II_1 factors.

Definition

The index of $A \subset B$ is $[B : A] := \dim_A(B)$.

Example

If R is the hyperfinite II_1 factor, and G is a finite group which acts outerly on R, then $R \subset R \rtimes G$ is a subfactor of index $|G|$.

If $H \leq G$, then $R \rtimes H \subset R \rtimes G$ is a subfactor of index $[G : H]$.

Theorem (Jones)

The possible indices for a subfactor are

$$\{4\cos\left(\frac{\pi}{n}\right)^2 | n \geq 3\} \cup [4, \infty].$$

Let $X = {}_AB_B$ and $\overline{X} = {}_BB_A$, and $\otimes = \otimes_A$ or $\otimes_B$ as needed.

Definition

The standard invariant of $A \subset B$ is the (planar) algebra of bimodules generated by X, denoted $\mathrm{Bim}(A, B)$*:*

$$\begin{array}{ccccccccc} X & , & X \otimes \overline{X} & , & X \otimes \overline{X} \otimes X & , & X \otimes \overline{X} \otimes X \otimes \overline{X} & , & \ldots \\ \overline{X} & , & \overline{X} \otimes X & , & \overline{X} \otimes X \otimes \overline{X} & , & \overline{X} \otimes X \otimes \overline{X} \otimes X & , & \ldots \end{array}$$

Definition

The principal graph of $A \subset B$ has vertices for (isomorphism classes of) irreducible A-A and A-B bimodules, and an edge from ${}_AY_A$ to ${}_AZ_B$ if $Z \subset Y \otimes X$ (iff $Y \subset Z \otimes \overline{X}$).

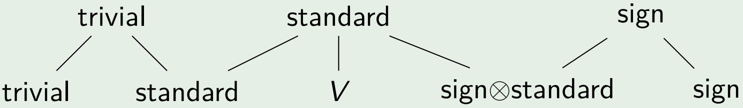
Ditto for the dual principal graph, with B-B and B-A bimodules.

Again, let G be a finite group with subgroup H, and act outerly on R. Consider $A = R \rtimes H \subset R \rtimes G = B$.

The irreducible B-B bimodules are of the form $R \otimes V$ where V is an irreducible G representation. The irreducible B-A bimodules are of the form $R \otimes W$ where W is an H irrep.

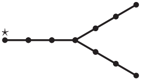
The dual principal graph of $A \subset B$ is the induction-restriction graph for irreps of H and G.

Example ($S_3 \leq S_4$)



(The principal graph is an induction-restriction graph too, for H and various subgroups of H.)

Some other principal graphs:

- Some Dynkin diagrams: A_n, D_{2n}, E_6, E_8
- All extended Dynkin diagrams: $A_n^{(1)}$, $D_n^{(1)}$, $E_n^{(1)}$, A_∞, $A_\infty^{(1)}$, D_∞.
- 

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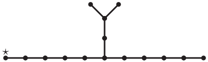
- 

- And more!

Definition

If the principal graphs of $A \subset B$ are finite, then we say $A \subset B$ is finite depth.

Question

How to construct examples (of subfactors, principal graphs, . . .)?

- Some come from other algebraic objects
- We leverage the ones we know
- Planar algebra constructions
- Connections on potential principal graphs

The combinatorial data of a subfactor is a connection:
The principal graphs can be assembled into a 4-partite graph:

$$\begin{tikzcd}
A - A \arrow[r, "-\otimes X"] \arrow[d, "\bar{X}\otimes -"] & A - B \arrow[d, "\bar{X}\otimes -"] \\
B - A \arrow[r, "-\otimes X"] & B - B
\end{tikzcd}$$

A connection assigns a number in $\mathbb{C}$ to any loop around this graph:

$$\begin{tikzcd}
P \arrow[r, dash, "e_1"] \arrow[d, dash, "e_4"] & Q \arrow[d, dash, "e_2"] \\
S \arrow[r, dash, "e_3"] & R
\end{tikzcd} = c(P, Q, R, S)$$

A connection assigns a number in $\mathbb{C}$ to any loop around the 4-partite principal graph:

$$\begin{tikzcd} P \arrow[r, dash, "e_1"] \arrow[d, dash, "e_4"] & Q \arrow[d, dash, "e_2"] \\ S \arrow[r, dash, "e_3"] & R \end{tikzcd} = c(P, Q, R, S)$$

This number is the coefficient of $(\bar{X} \otimes P) \otimes X \to S \otimes X \to R$ when $\bar{X} \otimes (P \otimes X) \to X \otimes Q \to R$ is written in the basis

$$(\bar{X} \otimes P) \otimes X \to ?? \otimes X \to R$$

So, $c(P, \cdot, R, \cdot)$ is a change-of-basis matrix for $\mathrm{Hom}(\bar{X} \otimes P \otimes X, R)$.

The connection $c(\cdot,\cdot,\cdot,\cdot)$ satisfies

- biunitarity:

$$c(P,\cdot,R,\cdot)\cdot\overline{c(P,\cdot,R,\cdot)} = Id$$

  and

$$c(\cdot,Q,\cdot,S)\cdot\overline{c(\cdot,Q,\cdot,S)} = Id$$

- renormalization:

$$c(P,Q,R,S) = \overline{c(Q,P,S,R)}\cdot\sqrt{\frac{d(Q)d(S)}{d(P)d(R)}}$$

- flatness: . . .

flatness: for all choices of P_i and Q_i,

$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[d, dash] & P_1 \arrow[r, dash] \arrow[d, dash] & P_2 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & 1 \arrow[d, dash] & \\
Q_1 \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & Q_1 \arrow[d, dash] & \\
Q_2 \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & Q_2 \arrow[d, dash] & = 1 \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
1 \arrow[r, dash] & P_1 \arrow[r, dash] & P_2 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 &
\end{tikzcd}$$

(Interpret this diagram as a sum over all possible edges and labels on the interior, and for each such labelling, take the product of the component unit squares.)

Theorem (Ocneanu)

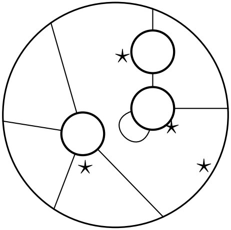
A flat biunitary connection on a pair of finite graphs gives a subfactor of the hyperfinite II_1 *factor with these principal graphs.*

Flatness can also be stated planar algebraically or ptensor categorically.

Definition (Jones)

A planar diagram has

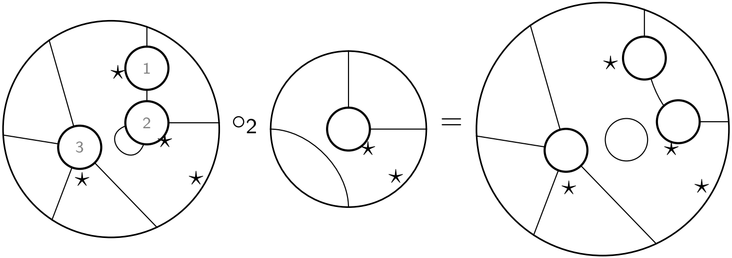
- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



In normal algebra (the kind with sets and functions), we have one dimension of composition:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

In planar algebras, we have two dimensions of composition



In abstract algebra, sets are given additional structure by functions. For example, a group is a set G with a multiplication law

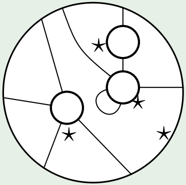
$$\circ : G \times G \to G.$$

A planar algebra also has sets, and maps giving them structure; there are a lot more of them.

Definition

A planar algebra is

- *a family of vector spaces* V_k, $k = 0, 1, 2, \ldots$, *and*
- *an interpretation of any planar diagram as a multi-linear map among* V_i*:*



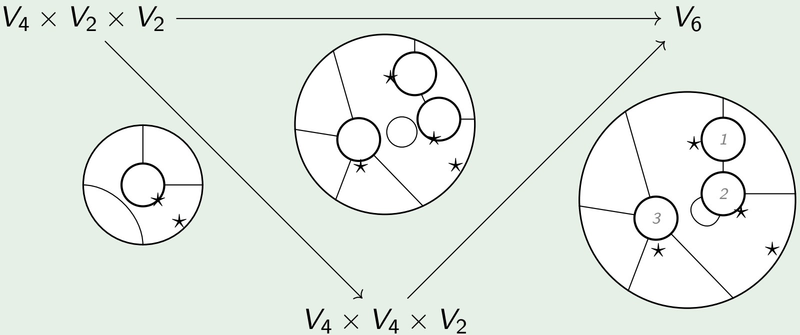
$$: V_2 \times V_5 \times V_4 \to V_7$$

Definition

A planar algebra is

- *a family of vector spaces V_k, $k = 0, 1, 2, \ldots$, and*
- *Planar diagrams giving multi-linear map among V_i,*

such that composition of multilinear maps, and composition of diagrams, agree:



$$\begin{tikzcd}
V_4 \times V_2 \times V_2 \arrow[rr] \arrow[dr] & & V_6 \\
& V_4 \times V_4 \times V_2 \arrow[ur] &
\end{tikzcd}$$

Definition

A Temperley-Lieb diagram is a way of connecting up points on the boundary of a circle labelled $1, \ldots, 2n$*, so that the connecting strings don't cross.*

For example, when $n = 3$*:* 

Example

The Temperley-Lieb planar algebra TL:

- The vector space TL_{2n} has a basis consisting of all Temperley-Lieb diagrams on $2n$ points.
- A planar diagram acts on Temperley-Lieb diagrams by placing the TL diagrams in the input disks, joining strings, and replacing closed loops of string by $\cdot\delta$.

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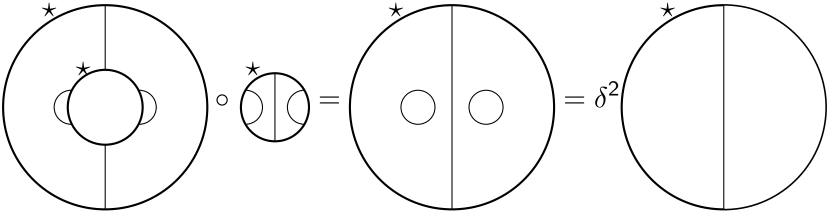


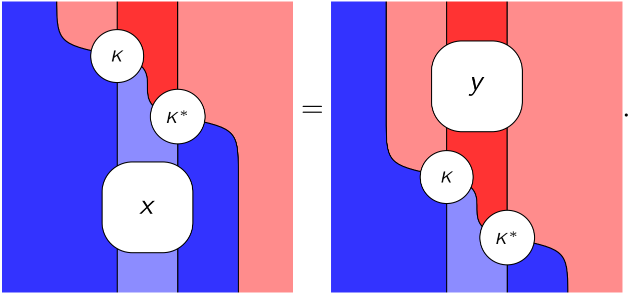
Example

The standard invariant of a subfactor (recall that this is built from those A-A, A-B, B-A and B-B bimodules tensor generated by $X = {}_A B_B$ and $\bar{X} = {}_B B_A$) is a planar algebra:

The vector space V_n is $\mathrm{Hom}(X^{\otimes n}, X^{\otimes n})$.

Action of planar tangles is built by labelling all strings by X or $\bar{X}$, interpreting concatenation as tensor product, and interpreting cups/caps using subfactor structure (inclusion, conditional expectation, multiplication).

Planar algebraic flatness: A connection is a biunitary 4-box in a graph planar algebra. An element x is flat if there is an element y such that



Definition

A monoidal category is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and

- *a unit object $\mathbf{1}$ of $\mathcal{C}$;*
- *unitor natural isomorphisms $\lambda_X : \mathbf{1} \otimes X \to X$ and $\rho_X : X \otimes \mathbf{1} \to X$;*
- *associator natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$;*

Definition

A monoidal category is $(\mathcal{C}, \otimes, \mathbf{1}, \lambda, \rho, \alpha)$ *satisfying*

- *(△) triangle equation :*

$$\begin{tikzcd}
(A \otimes \mathbf{1}) \otimes B \arrow[rr, "\alpha_{A,\mathbf{1},B}"] \arrow[dr, "\rho_A"] & & A \otimes (\mathbf{1} \otimes B) \arrow[dl, "\lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

- *(⬠) pentagon equation :*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[rr, "\alpha_{A,B,C} \otimes id_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A,B \otimes C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[dl, "id_A \otimes \alpha_{B,C,D}"] \\
& A \otimes (B \otimes (C \otimes D)) &
\end{tikzcd}$$

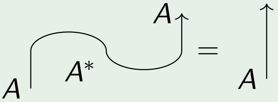
Definition

A ptensor category is a monoidal category $\mathcal{C}$ with additional structure:

- *For all objects C, D, $\mathrm{Hom}(C, D)$ is a finite dimensional $\mathbb{C}$-vector space.*
- *$\mathcal{C}$ is rigid: all objects have left, right duals: A, ${}^*A$, A^*, ${}^*A^{**}$, etc., and maps $ev : A \otimes A^* \to 1$, $coev : 1 \to A^* \otimes A$ satisfying the zig-zag requirement that*

$$\lambda_A \circ (ev \otimes id_A) \circ (id_A \otimes coev) \circ \rho_A^{-1} = id_A$$

– which turns into the picture



- *$\mathcal{C}$ is pivotal: there is a natural isomorphism $\psi : A \to A^{**}$.*
- *$\mathcal{C}$ is Karoubi complete, ie it is additively and idempotent closed.*

Example

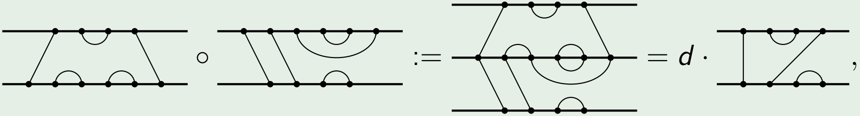
Among bimodules for a subfactor $A \subset B$ generated by $X =_A B_B$, the $A - A$ bimodules form a ptensor category (as do the $B - B$ bimodules):

- objects are bimodules,
- hom spaces are bimodule intertwiners,
- tensor is $\otimes_A$ (or $\otimes_B$),
- unit is ${}_A A_A$ (or ${}_B B_B$),
- duality is given by $*$.

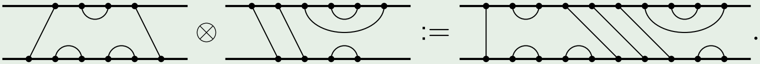
Example

The Temperley-Lieb-Jones ptensor category $TLJ(d)$:

- objects are finite strings of points on a horizontal line,
- hom spaces are connections among the points on two lines, modulo loop parameter d,
- Composition of morphisms is defined as vertical stacking, e.g.,



- The tensor operation is horizontal concatenation (of points on a line, or of diagrams), e.g.,



Example

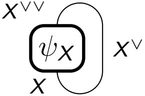
The graph category on a fixed set of vertices: $\mathrm{Graph}(V, d)$

- objects are oriented graphs on V,
- $\mathrm{Hom}(\Gamma, \Lambda)$ is given by linear maps $\mathbb{C}^{\Gamma(v\to w)} \to \mathbb{C}^{\Lambda(v\to w)}$ for all v, w. Composition is component-wise.
- The tensor operation is given by

$$(\Gamma \otimes \Lambda)(v \to w) := \sqcup_{u\in V}\Gamma(v \to u)\Lambda(u \to w).$$

- The unit of $\mathrm{Graph}(V, d)$ consists of a disjoint union of loops, one for each $v \in V$.
- And the dual of Γ is the same graph with its edges reversed.

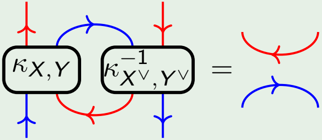
The universal property of $TLJ(d)$ is that, if $\mathcal{C}$ is a ptensor category, giving a pivotal ptensor functor $TLJ(d) \to \mathcal{C}$ is equivalent to choosing an object X of $\mathcal{C}$ with the property that



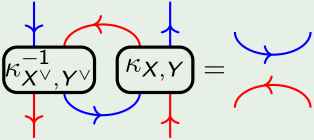
$= d \cdot \mathbf{1}_{1_\mathcal{C}}$

Definition

A connection is a pair of functors $F, G : TLJ(d) \to \mathrm{Graph}(V, d)$ *and a commutator natural isomorphism* $\kappa_{c,d} : F(c) \otimes G(d) \to G(d) \otimes F(c)$*, which is bi-invertible:*



and



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Recall that flatness of a connection means that for all choices of P_i and Q_i,

$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[d, dash] & P_1 \arrow[r, dash] \arrow[d, dash] & P_2 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & 1 \arrow[d, dash] & \\
Q_1 \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & Q_1 \arrow[d, dash] & \\
Q_2 \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & Q_2 \arrow[d, dash] & = 1 \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
1 \arrow[r, dash] & P_1 \arrow[r, dash] & P_2 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 &
\end{tikzcd}$$

(Interpret this diagram as a sum over all possible edges and labels on the interior, and for each such labelling, take the product of the component unit squares.)

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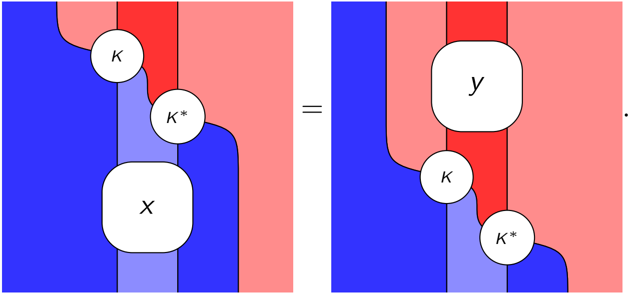

Suppose $\mathcal{M}$ is a semisimiple $TLJ(d) - TLJ(d)$ bimodule category. The categories $\mathrm{End}(\mathcal{M})$ and $\mathrm{Graph}(V, d)$ are equivalent if $\# V$ is the number of isomorphism classes of simples of M.

Given m, is it possible to put a compatible ptensor structure with unit m on $\mathcal{M}$? Define the left flat part $\mathcal{L}$ of $\mathcal{M}$ to be the dominant image of TLJ in $\mathrm{End}_{TLJ}(\mathcal{M})$.

$$\begin{tikzcd}
TLJ \arrow[r, two heads, "F"] \arrow[drrr, "- \triangleright m"'] & \mathcal{L} \arrow[r, hook] & \mathrm{End}_{TLJ}(\mathcal{M}) \arrow[r, hook] & \mathrm{End}(\mathcal{M}) \arrow[d, "\mathrm{ev}_m"] & \mathrm{End}_{TLJ}(\mathcal{M}) \arrow[l, hook'] & \mathcal{R} \arrow[l, hook'] & TLJ \arrow[l, two heads, "G"'] \arrow[dlll, "m \triangleleft -"] \\
 & & & \mathcal{M} & & &
\end{tikzcd}$$

Consider the composite map $\mathcal{L} \to \mathcal{M}$, still denoted ev_m, which fits into the following commutative diagram.

$$\begin{tikzcd}
TLJ \arrow[r, two heads] \arrow[dr, "- \triangleright m"'] & \mathcal{L} \arrow[d, "\mathrm{ev}_m"] \\
 & \mathcal{M}.
\end{tikzcd}$$

$$\begin{tikzcd}
TLJ \arrow[r, two heads, "F"] \arrow[drrr, "-\triangleright m"'] & \mathcal{L} \arrow[r, hook] & \mathrm{End}_{TLJ}(\mathcal{M}) \arrow[r, hook] & \mathrm{End}(\mathcal{M}) \arrow[d, "\mathrm{ev}_m"] & \mathrm{End}_{TLJ}(\mathcal{M}) \arrow[l, hook'] & \mathcal{R} \arrow[l, hook'] & TLJ \arrow[l, two heads, "G"'] \arrow[dlll, "m \triangleleft -"] \\
 & & & \mathcal{M} & & &
\end{tikzcd}$$

If the evaluation map $\mathcal{L} \to \mathcal{M}$ is full, there is a splitting $\mathcal{M} \to \mathcal{L}$:

$$\begin{tikzcd}
TLJ \arrow[r, two heads] \arrow[dr, "-\triangleright m"'] & \mathcal{L} \arrow[d, shift left, "\mathrm{ev}_m"] \\
 & \mathcal{M}. \arrow[u, dashed, shift left]
\end{tikzcd}$$

A splitting gives an equivalence between $\mathcal{L}$ and $\mathcal{M}$ and allows us to import the tensor structure of $\mathcal{L}$ to $\mathcal{M}$. So if such a splitting exists, we say $\mathcal{M}$ is a flat $TLJ - TLJ$ bimodule.

The End!