T-Shifts on Reduced Plabic Graphs and Legendrian Weaves

Daping Weng

UC Davis

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Joint work with Roger Casals, Ian Le, and Melissa Sherman-Bennett

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Table of Contents

1 Overview

2 Positroid Strata and Reduced Plabic Graphs

3 Legendrian Weaves

4 Legendrian Weaves from Reduced Plabic Graphs

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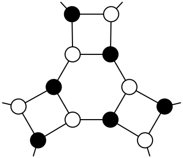

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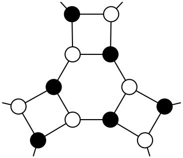

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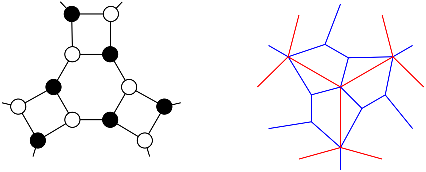

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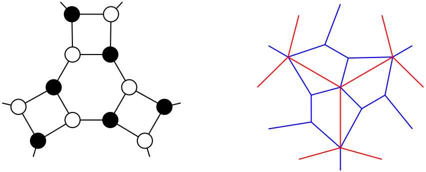

- However, one can only realize certain mutations as moves on reduced plabic graphs.
- On the other hand, the recent introduction and development of Legendrian weaves [CZ22, CW22, CGG$^+$22] has built an explicit connection between cluster structures and geometry.
- The initial goal of our project is to construct weaves from reduced plabic graphs, which gives rise to a geometric interpretation of the cluster structures on positroid strata. As an application, we prove the conjecture of the equality between the Muller-Speyer twist map and the Donaldson-Thomas transformation.

Positroid Strata and Reduced Plabic Graphs

Definition of Positroids

Positroid strata come from a stratification of $\mathrm{Gr}_{m,n}^{\geq 0}(\mathbb{R})$ [Pos06]. For $m \leq n$,

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$$\mathrm{Gr}_{3,6}(\mathbb{R}) = \mathrm{GL}_3(\mathbb{R}) \Bigg\backslash \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix}$$

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$$\Delta_{2,3,5} = \det \begin{pmatrix} x_{12} & x_{13} & x_{15} \\ x_{22} & x_{23} & x_{25} \\ x_{32} & x_{33} & x_{35} \end{pmatrix}.$$

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Definition

$\mathcal{P}$ is called a **positroid** of type (m, n) if the subset $C_{\mathcal{P}} \subset \mathrm{Gr}^{\geq 0}_{m,n}(\mathbb{R})$ is non-empty. We call m the **rank** of $\mathcal{P}$.

Grassmann Necklace and Positroid Stratum

For each $i \in [n] := \{1, 2, \ldots, n\}$, we define a linear order $<_i$ on $[n]$ by

$$i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1.$$

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Definition ([KLS13])

Let $\mathcal{P} \subset \binom{[n]}{m}$ be a positroid and let $\mathcal{I}_{\mathcal{P}} = (I_1, \ldots, I_n)$ be its Grassmann necklace. The **(open) positroid stratum** $\Pi^\circ_{\mathcal{P}}$ is defined to be the following subvariety of $\widetilde{\mathrm{Gr}}_{m,n}$:[1]

$$\Pi^\circ_{\mathcal{P}} = \left\{ [M] \in \widetilde{\mathrm{Gr}}_{m,n} \;\middle|\; \prod_i \Delta_{I_i}(M) \neq 0 \text{ and } \Delta_I([M]) = 0 \text{ for all } I \notin \mathcal{P} \right\}.$$

[1] Here $\widetilde{\mathrm{Gr}}_{m,n}$ denotes the affine cone of the Grassmannian $\mathrm{Gr}_{m,n}$ with respect to the Plücker embedding.

Positroid Legendrian Link

Given a Grassmann necklace $\mathcal{I}_{\mathcal{P}} = (I_1, I_2, \ldots, I_n)$, we can draw a **Legendrian link**[2] $\Lambda_{\mathcal{P}}$ by

[2] A Legendrian link is a link in $\mathbb{R}^3_{x,y,z}$ satisfying $y = dz/dx$. Here we are describing the Legendrian link as a satellite of the max tb unknot.

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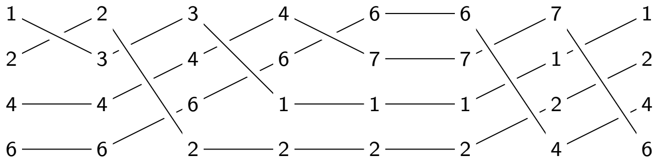

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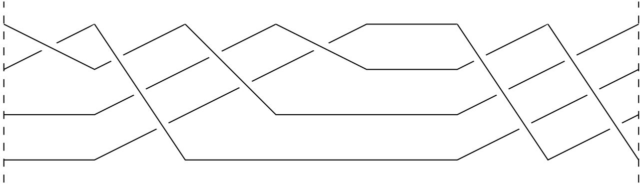

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Bounded Affine Permutations

In a Grassmann necklace $\mathcal{I}_{\mathcal{P}} = (I_1, \ldots, I_n)$, any two cyclically consecutive entries differ at most by one element. Thus, we define the **bounded affine permutation** f of $\mathcal{P}$ by

$$f_{\mathcal{P}}(i) = \begin{cases} j & \text{if } i \in I_i \text{ and } I_{i+1} = (I_i \setminus \{i\}) \cup \{j\} \text{ for } j > i, \\ j+n & \text{if } i \in I_i \text{ and } I_{i+1} = (I_i \setminus \{i\}) \cup \{j\} \text{ for } j \leq i, \\ i & \text{if } i \notin I_i. \end{cases}$$

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One can also think of $f_{\mathcal{P}}$ as searching for the first column vector $v_{f_{\mathcal{P}}(i)}$ to the right of v_i (cyclically) in $[M]$ such that v_i is in the span of $\{v_{i+1}, \dots, v_{f_{\mathcal{P}}(i)}\}$. For example,

$$M = \begin{pmatrix} 1 & 0 & * & * & 1 & 1 \\ 0 & 1 & * & * & 1 & 0 \\ 0 & 0 & * & * & 0 & 0 \end{pmatrix} \qquad f_{\mathcal{P}}(5) = 2 + 6 = 8$$

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Theorem (Postnikov [Pos06])

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$$m = \frac{1}{n} \sum_{i=1}^{n} (f_\mathcal{P}(i) - i).$$

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For example, the top dimensional positroid stratum in $\mathrm{Gr}_{m,n}$ corresponds to the bounded affine permutation $f_\mathcal{P}(i) = i + m$.

Reduced Plabic Graphs

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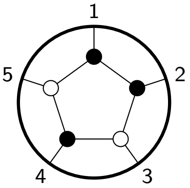

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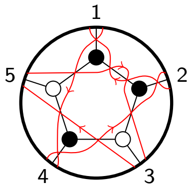

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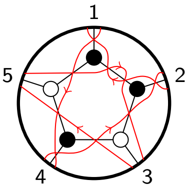
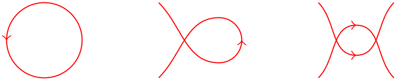
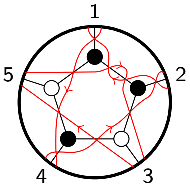


A plabic graph is said to be **reduced** if its zig-zags do not form loops, self-intersection, or parallel bigons.



Each positroid stratum $\Pi^\circ_{\mathcal{P}}$ is a cluster $\mathcal{A}$-variety. An initial cluster seed of $\Pi^\circ_{\mathcal{P}}$ can be constructed from a **reduced plabic graph**.

A **plabic graph** $\mathbb{G}$ is a planar graph on a disk $\mathbb{D}$ with **solid** and **empty** vertices and external edges attached to $\partial\mathbb{D}$. We label the external edges by $1, 2, \ldots, n$. By following "rules of the road", we can draw **zig-zags** along edges of $\mathbb{G}$.



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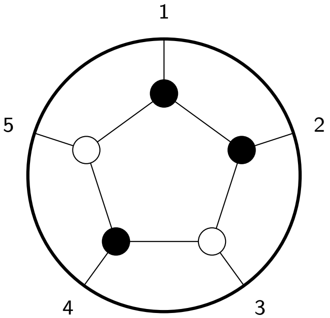
A reduced plabic graph $\mathbb{G}$ is associated with a positroid $\mathcal{P}$ if the zig-zags in $\mathbb{G}$ go from i to $f_{\mathcal{P}}(i) \mod n$ for each i.

Initial Cluster Seed

Let $\mathbb{G}$ be a reduced plabic graph for a positroid $\mathcal{P}$ of type (m, n). To each face F of $\mathbb{G}$, we can associate an m-element set I_F by using the targets of zig-zags (also known as the **target labeling**).



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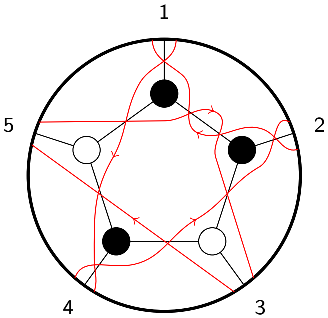

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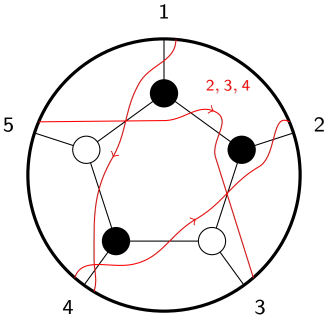

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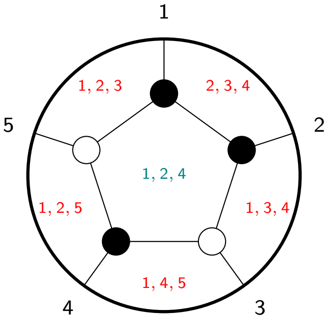

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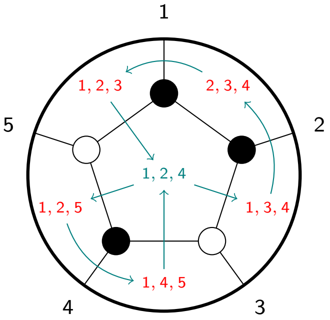

We can then draw a collection of arrows across edges such that they form a counterclockwise cycle around solid vertices. This defines a quiver Q with frozen vertices. The pair $(\{\Delta_{I_F}\}_F, Q)$ defines an initial seed for the cluster structure on $\Pi^{\circ}_{\mathcal{P}}$.

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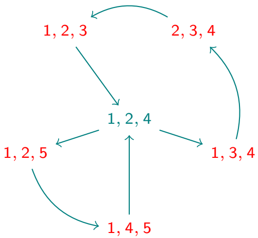

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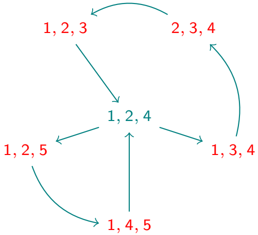

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Here is a javascript program that can generate a Le plabic graph together with the initial cluster seed for a positroid.

Legendrian Weaves

Definition

Legendrian weaves were introduced by Casals and Zaslow [CZ22]. A **Legendrian** m**-weave** $\mathfrak{w}$ is a graph on $\mathbb{D}$ with colored edges special vertices, and it describes the singular locus of certain immersed surface $\Sigma_{\mathfrak{w}}$ that is a generic $m : 1$ cover of $\mathbb{D}$.

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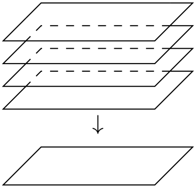

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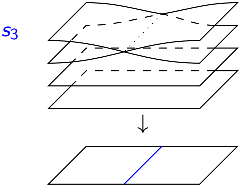

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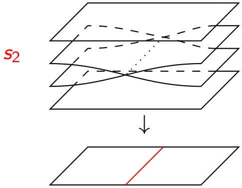

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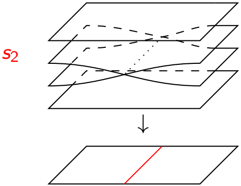

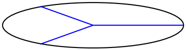

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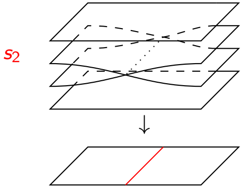

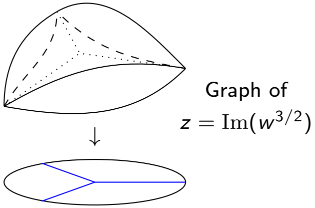

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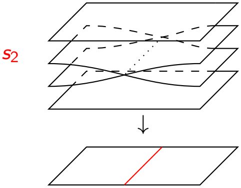
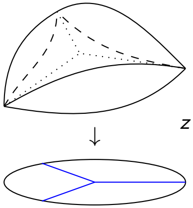
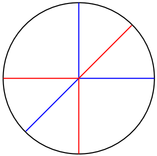




Graph of
$z = \mathrm{Im}(w^{3/2})$



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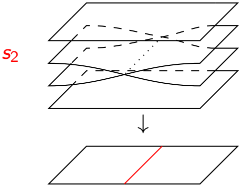

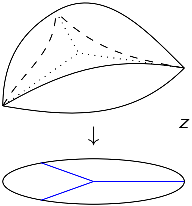

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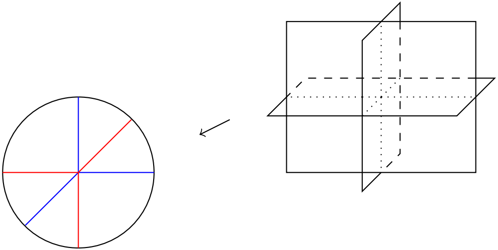

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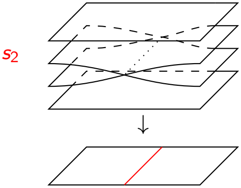

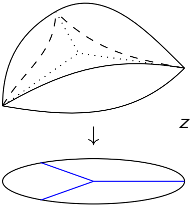

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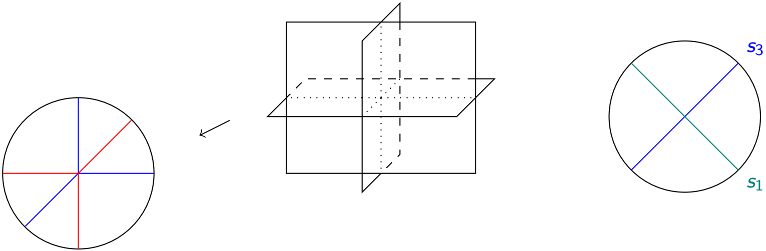

Exact Lagrangian Surface

Let x_1 and x_2 be coordinates on $\mathbb{D}$ and let z be the height coordinate for $\Sigma_{\mathfrak{w}}$. By setting $y_i := \frac{\partial z}{\partial x_i}$, we get an exact Lagrangian surface $L_{\mathfrak{w}}$ in $T^*\mathbb{D} \cong \mathbb{R}^4_{x_1,x_2,y_1,y_2}$. In particular, $L_{\mathfrak{w}}$ is diffeomorphic to the m-fold **spectral cover** of $\mathbb{D}$.

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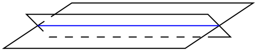

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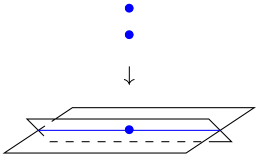

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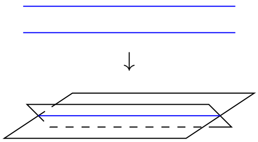

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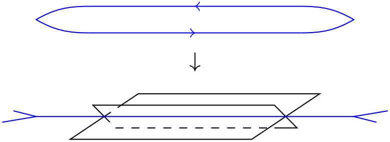
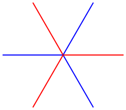
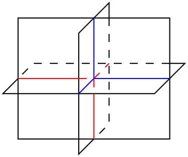


Y-cycles have two possible behaviors at a hexavalent weave vertex:





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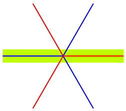

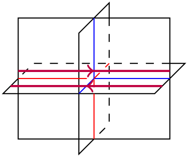

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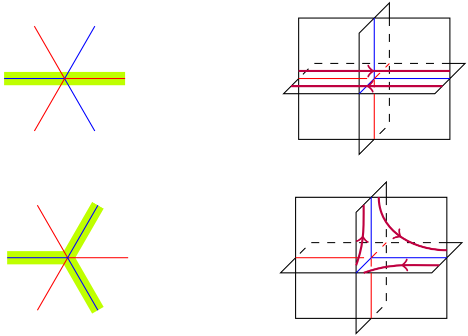

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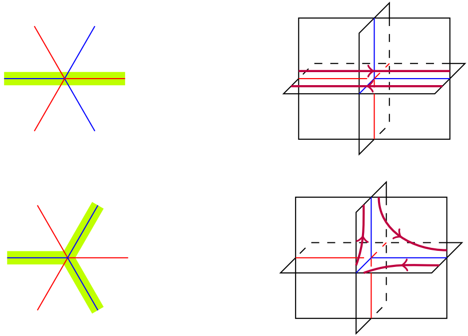

The intersection pairing between these 1-cycles can be computed by summing up local contributions.

Cluster Ensemble from Legendrian Weaves

In [CW22], we lay down the framework of describing a cluster ensemble structure using Legendrian weaves.

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quiver vertices	$\longleftrightarrow$	Y-cycles on $L_{\mathfrak{w}}$
quiver arrows	$\longleftrightarrow$	intersection pairing between Y-cycles
cluster $\mathcal{X}$-torus	$\longleftrightarrow$	rank 1 local system on $L_{\mathfrak{w}}$
cluster $\mathcal{X}$-variables	$\longleftrightarrow$	monodromies along Y-cycles
cluster $\mathcal{A}$-torus	$\longleftrightarrow$	decorated rank 1 local system on $L_{\mathfrak{w}}$
cluster $\mathcal{A}$-variables	$\longleftrightarrow$	parallel transports along relative 1-cycles that are Poincaré dual to the Y-cycles
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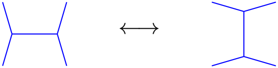

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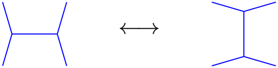

Following [CZ22], the (decorated) rank 1 local system on $L_{\mathfrak{w}}$ can be described as certain flag moduli space on $\mathbb{D}$.

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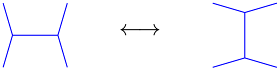
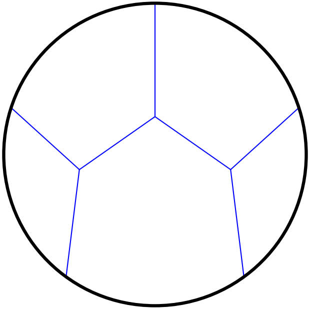


Following [CZ22], the (decorated) rank 1 local system on $L_{\mathfrak{w}}$ can be described as certain flag moduli space on $\mathbb{D}$.

It was also proved in [CW22] that if a Y-cycle is given by a tree on $\mathfrak{w}$ (also known as a **Y-tree**), then the new exact Lagrangian surface from Polterovich surgery can again be described by Legendrian weaves.

Example: A_2 Cluster Ensemble



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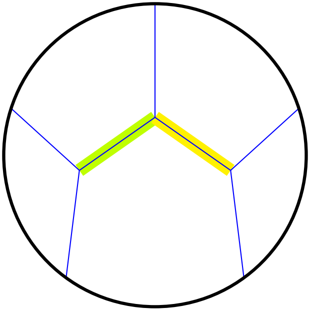

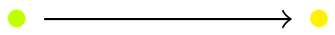

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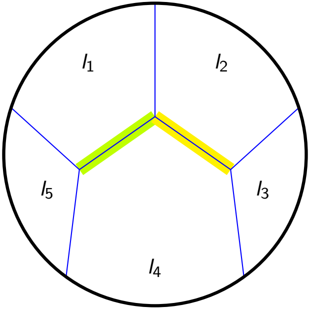

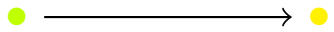

Legendrian Weaves from Reduced Plabic Graphs

When studying $m = 2$ amplituhedron, Parisi, Sherman-Bennett, and Williams [PSBW21] develop a procedure that produces a new reduced plabic graph $\mathbb{G}^{\downarrow}$ of rank $m-1$ from a reduced plabic graph $\mathbb{G}$ of rank m. We call this procedure a **T-shift**[3].

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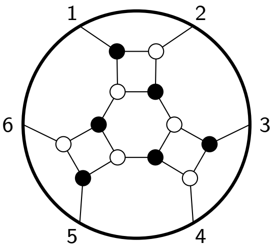

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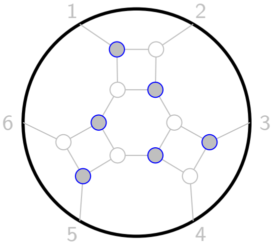

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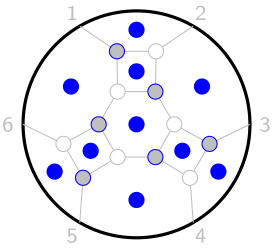

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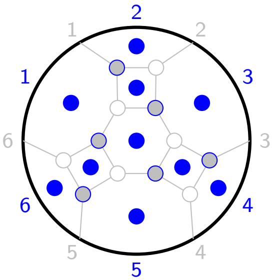

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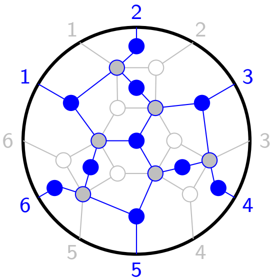

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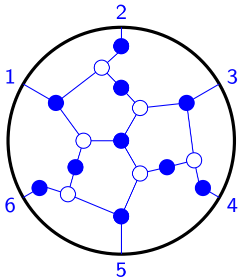

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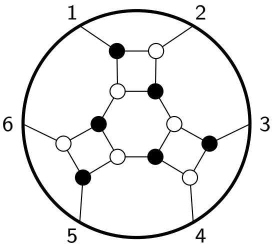

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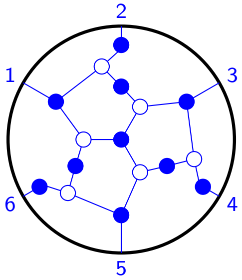

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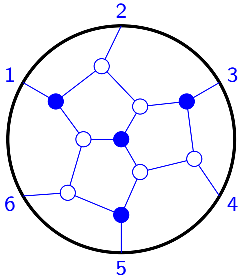

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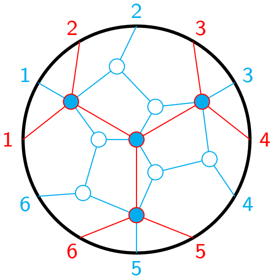

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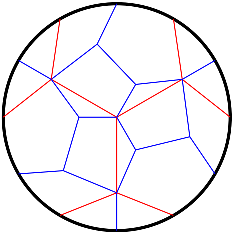

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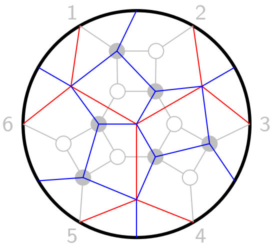

Flag Moduli Space

The flag moduli space associated with a positroid weave $\mathfrak{w}$ associates a flag with each connected component of the complement of $\mathfrak{w}$. We can recover these flags by going backward along the iterative T-shifts.

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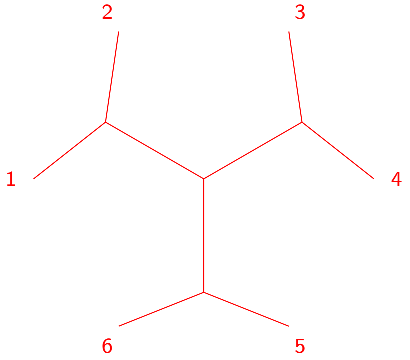

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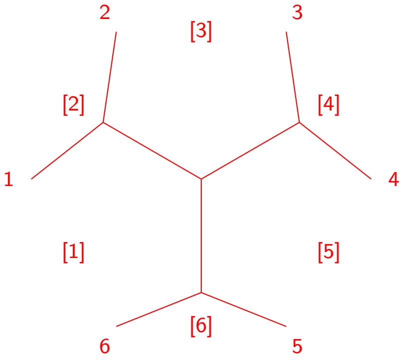

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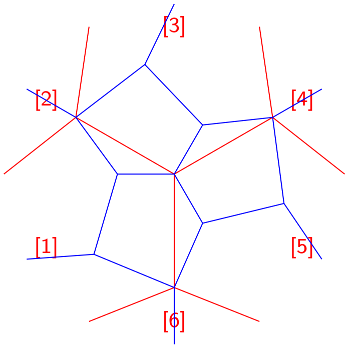

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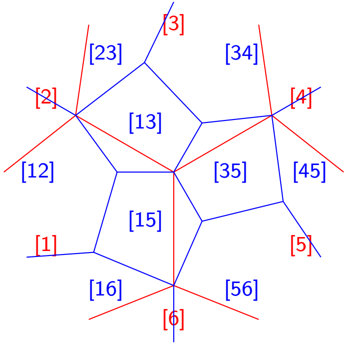

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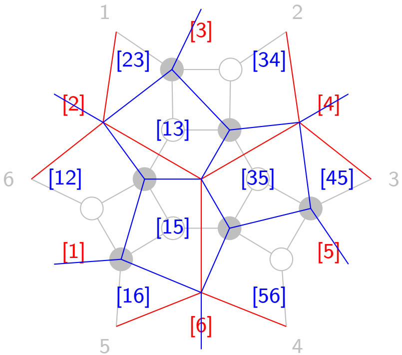

Note that we can recover the target labeling of the faces of $\mathbb{G}$ by repeating this backward iteration once more.

Mutation at Non-Square Faces

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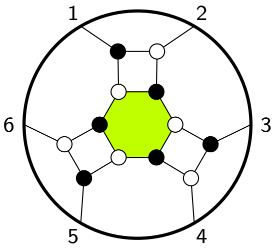

$\xrightarrow{\text{Mutate}}$???

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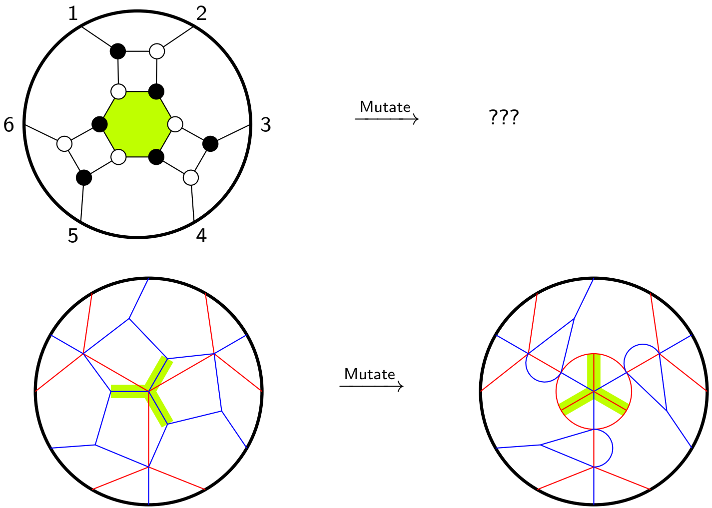

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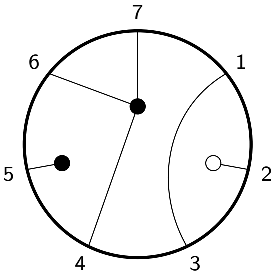

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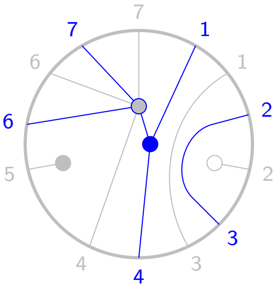

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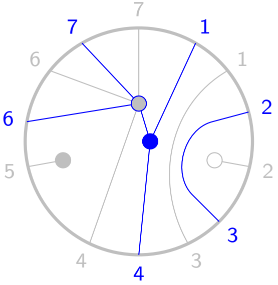

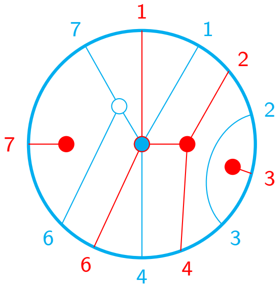

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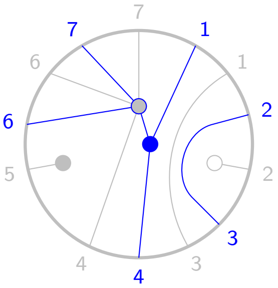

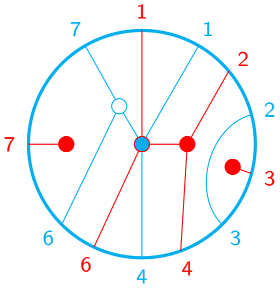

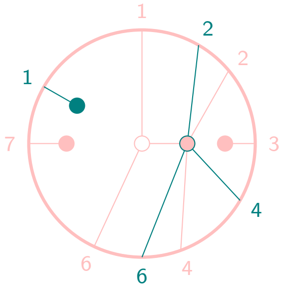

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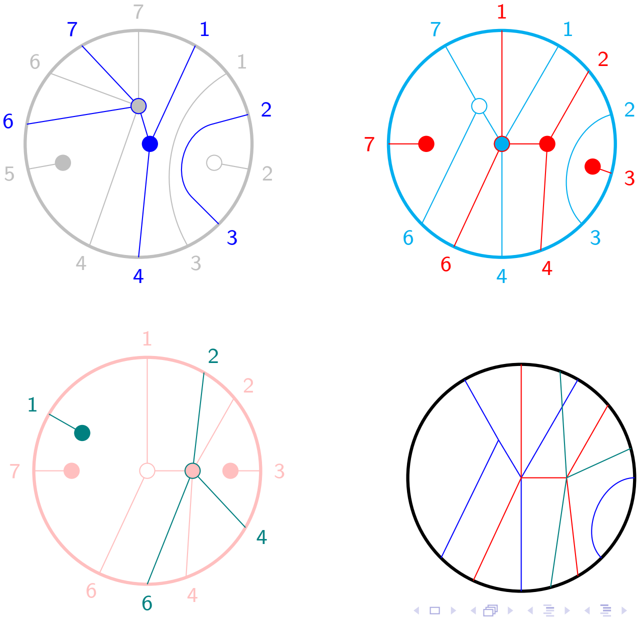

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The boundary of a positroid weave $\mathfrak{w}$ is $\Lambda_{\mathcal{P}}$ (in other words, $L_{\mathfrak{w}}$ is an exact Lagrangian filling of $\Lambda_{\mathcal{P}}$). Moreover, $\Pi^{\circ}_{\mathcal{P}} \cong \mathfrak{M}(\Lambda_{\mathcal{P}}; \mathfrak{t})$, where $\Lambda_{\mathcal{P}}$ is the positroid Legendrian link associated with $\mathcal{P}$ and $\mathfrak{t}$ is a certain way to decorate $\Lambda_{\mathcal{P}}$ with base points.

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In [GK13], Goncharov and Kenyon construct a **conjugate surface** $S_{\mathbb{G}}$ from a reduced plabic graph $\mathbb{G}$. Shende, Treumann, Williams, and Zaslow [STWZ19] prove that the conjugate surface $S_{\mathbb{G}}$ can be seen as an exact Lagrangian surface in $T^*\mathbb{D}$.

Theorem

The iterative T-shift procedure can be viewed geometrically as a Hamiltonian isotopy from $S_{\mathbb{G}}$ to $L_{\mathfrak{w}}$.

More Implications

In [CW22] and [CGG$^+$22], the **Donaldson-Thomas transformation** of the cluster ensemble can be geometrically described as the composition of a Legendrian isotopy that cyclically rotates $\Lambda_{\mathcal{P}}$ and a contactomorphism that is analogous to a transposition. By comparing this action with the Muller-Speyer twist map [MS17] on positroid strata, we prove that the following conjecture.

Theorem

The Muller-Speyer twist map on a positroid stratum coincides with its (quasi-cluster) Donaldson-Thomas transformation.

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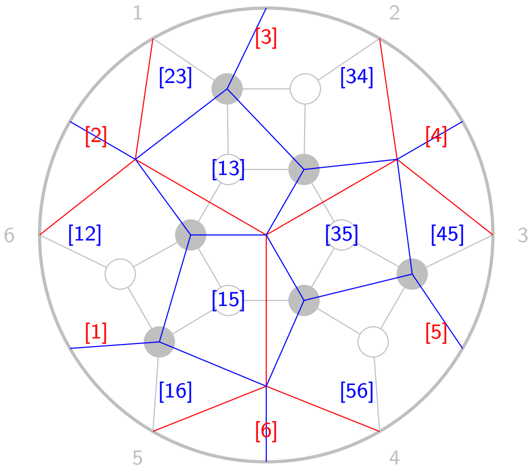
Recently, Decampo and Muller [DM23] introduce a moduli space of **linear recurrence** with a cluster $\mathcal{X}$ structure for each positroid. These moduli spaces can be thought of as the "mirror" of positroid strata. We can modify our construction and obtain a different decorated flag moduli space with the same cluster $\mathcal{X}$ structure as loc. cit.

Thank You!



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