Monoidal Triangular Geometry with Applications to Representation Theory

(joint work with Kent Vashaw and Milen Yakimov)

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NVY-References

[NVY22a] D.K. Nakano, K.B. Vashaw, M.T. Yakimov, *Noncommutative tensor triangular geometry*, American J. Math., 144, (2022), 1-44.

[NVY22b] D.K. Nakano, K.B. Vashaw, M.T. Yakimov, *Noncommutative tensor triangular geometry and the tensor product property for support maps*, IMRN, 22, (2022), 17766-17796.

[NVY23] D.K. Nakano, K.B. Vashaw, M.T. Yakimov, *On the spectrum and support theory of finite tensor categories*, ArXiv 2112.11170.

Complexity

Definition (Alperin, 1977)

Let A be a finite-dimensional k-algebra, and $M \in \text{mod}(A)$. Let $P_\bullet \twoheadrightarrow M$ be a minimal projective resolution for M.

Define the *complexity of* M, $c_A(M) = s$, to be the rate of growth of this projective resolution, $r(\{P_n : n = 0, 1, 2, \ldots\})$, (i.e., smallest integer $s \geq 0$ such that $\dim P_n \leq Cn^{s-1}$ for some $C > 0$)

Note that if A is self-injective then $c_A(M) = 0$ if and only if M is a projective A-module.

Example

Let $\mathfrak{g} = \mathfrak{sl}(2)$ and $A = u(\mathfrak{g})$. The simple modules in the principal block are indexed by $L(0)$ and $L(p-2)$. The projective cover $P(\lambda)$ of $L(\lambda)$ is $2p$-dimensional for $\lambda = 0, p-2$.

The minimal projective resolution of the trivial module $L(0)$ is given by

$$\cdots \to P(p-2) \oplus P(p-2) \to P(0) \to L(0) \to 0.$$

Therefore, $\dim P_n = (2p)(n+1)$ and $c_{u(\mathfrak{g})}(L(0)) = 2$ (rate of growth of the minimal proj. resolution). In fact, one can also show that $c_{u(\mathfrak{g})}(L(p-2)) = 2$.

Finite Generation of Cohomology and Support Varieties

Let A be a finite-dimensional Hopf algebra over k (e.g., $A = kG$ where G is a finite group). Set $R = \mathrm{H}^\bullet(A, k)$ cohomology ring and assume that R is finitely generated and $\mathrm{Ext}_A^\bullet(M, N)$ is a finitely generated R-module for all $M, N \in \mathrm{mod}(A)$.

Definition (Carlson et. al. 1980's)

The *support variety* of M:

$$V_A(M) = \{P \in \mathrm{Spec}(\mathrm{H}^\bullet(A, k)) : \ \mathrm{Ext}_A^\bullet(M, M)_P \neq 0\}.$$

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- $c_A(M) = \dim V_A(M)$

- The finite generation for arbitrary finite tensor categories is an open conjecture (Etingof-Ostrik). For cocommutative Hopf algebra, this was established by Friedlander-Suslin (1995).

A Lie Theoretic Example: Jantzen's Conjecture

Let G be a simple, simply connected algebraic group (scheme) defined over $\mathbb{F}_p$, $F : G \to G$ the Frobenius morphism, and $G_1 = \ker F$.

- $R = \mathrm{H}^\bullet(G_1, k) \cong \mathrm{H}^\bullet(u(\mathfrak{g}), k)$ is finitely generated.
- $V_{G_1}(k) \cong \{x \in \mathfrak{g} : \ x^{[p]} = 0\}$ (support variety of the trivial module)
- $V_{G_1}(M) \subseteq \mathcal{N}$ where $\mathcal{N}$ is the nilpotent cone for $M \in \mathrm{mod}(G_1)$.

Theorem (Nakano, Parshall, Vella (2002))

Let G be a simple simply connected algebraic group and assume that p is good. Let $\lambda \in X(T)_+$. Choose $w \in W$ such that $w(\Phi_{\lambda,p}) = \Phi_J$ for some $J \subseteq \Delta$. Then $V_{G_1}(\nabla(\lambda)) = G \cdot \mathfrak{u}_J$ where $\nabla(\lambda) = \mathcal{H}^0(G/B, \mathcal{L}(\lambda))$.

Hence, $c_{G_1}(\nabla(\lambda)) = 2 \dim \mathfrak{u}_J$.

“Categories Have Hidden Geometry”

Given a symmetric tensor triangular category, $\mathbf{K}$, Balmer (2005) first introduced the notion of the categorical spectrum $\mathrm{Spc}(\mathbf{K})$ by defining (completely) prime ideals in $\mathbf{K}$ via object-wise tensoring. These ideas are known as *tensor triangular geometry*.

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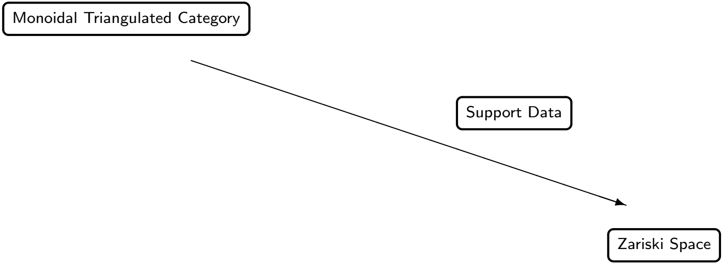

Monoidal Triangulated Categories [NVY]

Definition

A *monoidal triangulated category* (MΔC) is a triple $(\mathbf{K}, \otimes, \mathbf{1})$ such that

(i) $\mathbf{K}$ is a triangulated category,

(ii) $\mathbf{K}$ has a monodial tensor product $\otimes : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ which is exact in each variable with unit object $\mathbf{1}$.

Examples:

Example

Let A be a finite-dimensional Hopf algebra. Then

(iii) $\mathbf{K}^c = \operatorname{stmod}(A)$ stable module category of finite-dimensional modules for A

(iv) $\mathbf{K} = \operatorname{StMod}(A)$ stable module category for $\operatorname{Mod}(A)$.

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Example

Let R be a commutative Noetherian ring. Let

(i) $\mathbf{K}^c = D^b_{perf}(R)$ bounded derived category of finitely generated projective R-modules

(ii) $\mathbf{K} = D(R)$ derived category of R-modules.

Then $\mathbf{K}^c$ and $\mathbf{K}$ are tensor triangulated categories.

"Treat a M∆C Like a Ring"

Definition

(a) A *(tensor) ideal* in $\mathbf{K}$ is a triangulated subcategory $\mathbf{I}$ of $\mathbf{K}$ such that $M \otimes N \in \mathbf{I}$ and $N \otimes M \in \mathbf{I}$ for all $M \in \mathbf{I}$ and $N \in \mathbf{K}$.

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(c) A *completely prime ideal* $\mathbf{P}$ of $\mathbf{K}$ is a proper thick tensor ideal such that if $M \otimes N \in \mathbf{P}$ then either $M \in \mathbf{P}$ or $N \in \mathbf{P}$.

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(d) [NEW] A *prime ideal* $\mathbf{P}$ of $\mathbf{K}$ is a proper thick tensor ideal such that $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$ implies that $\mathbf{I} \subseteq \mathbf{P}$ or $\mathbf{J} \subseteq \mathbf{P}$ for all thick ideals $\mathbf{I}$, $\mathbf{J}$ of $\mathbf{K}$. .

A Generalization of Paul Balmer's Categorical Spectrum

Definition

The *Balmer spectrum* is defined as

$$\mathrm{Spc}(\mathbf{K}) = \{\mathbf{P} \subset \mathbf{K} \mid \mathbf{P} \text{ is a prime ideal}\}.$$

The topology on $\mathrm{Spc}(\mathbf{K})$ is given by closed sets of the form

$$Z(\mathcal{C}) = \{\mathbf{P} \in \mathrm{Spc}(\mathbf{K}) \mid \mathcal{C} \cap \mathbf{P} = \varnothing\}$$

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One can also define

$$\text{CP-Spc}(\mathbf{K}) = \{\mathbf{P} \subset \mathbf{K} \mid \mathbf{P} \text{ is a completely prime ideal}\}.$$

$$\text{CP-Spc}(\mathbf{K}) \subseteq \mathrm{Spc}(\mathbf{K}).$$

Zariski Spaces

Definition

Assume throughout that X is a Noetherian topological space. In this case any closed set in X is the union of finitely many irreducible closed sets. We say that X is a *Zariski space* if in addition any irreducible closed set Y of X has a unique generic point (i.e., $y \in Y$ such that $Y = \overline{\{y\}}$).

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Example

Let R is a commutative Noetherian ring.

(1) $X = \text{Spec}(R)$.

(2) $X = \text{Proj}(\text{Spec}(R)) := \text{Proj}(R)$ if R is graded.

(3) $X = G\text{-Proj}(R)$ if R is graded and G is an algebraic group.

Zariski Spaces: Notation

(ii) $\mathcal{X}$ be the collection of subsets of X.

(ii) $\mathcal{X}_{cl}$ be the collection of closed subsets of X.

(iii) $\mathcal{X}_{irr}$ be the set of irreducible closed sets.

(iv) A subset W in X is *specialization closed* if and only if $W = \cup_{j \in J} W_j$ where W_j are closed sets.

(v) $\mathcal{X}_{sp}$ be the collection of all specialization closed subsets of X.

Support Data

Definition

A *support data* (resp. *weak support data*) is an assignment $\sigma : \mathbf{K} \to \mathcal{X}$ which satisfies the following six properties (for $M, M_i, N, Q \in \mathbf{K}$):

(S1) $\sigma(0) = \varnothing$, $\sigma(\mathbf{1}) = X$;

(S2) $\sigma(\oplus_{i\in I} M_i) = \bigcup_{i\in I} \sigma(M_i)$ whenever $\oplus_{i\in I} M_i$ is an object of $\mathbf{K}$;

(S3) $\sigma(\Sigma M) = \sigma(M)$;

(S4) for any distinguished triangle $M \to N \to Q \to \Sigma M$ we have

$$\sigma(N) \subseteq \sigma(M) \cup \sigma(Q);$$

(S5) $\bigcup_{C\in\mathbf{K}} \sigma(M \otimes C \otimes N) = \sigma(M) \cap \sigma(N)$;

(WS5) $\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$, $\mathbf{I}$ and $\mathbf{J}$ are ideals of $\mathbf{K}$: $\Phi_\sigma(\mathbf{I}) = \cup_{A\in\mathbf{I}}\sigma(A)$.

(S6) $\sigma(M) = \sigma(M^*)$ for $M \in \mathbf{K}^c$ [the compact objects have a duality].

We will be interested in support data which satisfy an additional two properties:

(S7) $\sigma(M) = \varnothing$ if and only if $M = 0$; (Faithfulness Property)

(S8) for any $W \in \mathcal{X}_{cl}$ there exists an $M \in \mathbf{K}^c$ such that $\sigma(M) = W$ (Realization Property).

Classifying Thick Tensor Ideals and the Balmer Spectrum

Theorem (BKN16, Dell'Ambrogio, NVY19)

Let $\mathbf{K}$ *be a compactly generated* $M\Delta C$*. Let* X *be a Zariski space and let* $\sigma : \mathbf{K} \to \mathcal{X}$ *be a weak support data satisfying the additional conditions (S7) and (S8) with* $\sigma(\langle M \rangle) \in \mathcal{X}_{cl}$ *for* $M \in \mathbf{K}^c$*. There is a pair of mutually inverse maps*

$$\{\textit{thick tensor ideals of } \mathbf{K}^c\} \underset{\Theta}{\overset{\Phi_\sigma}{\rightleftarrows}} \mathcal{X}_{sp},$$

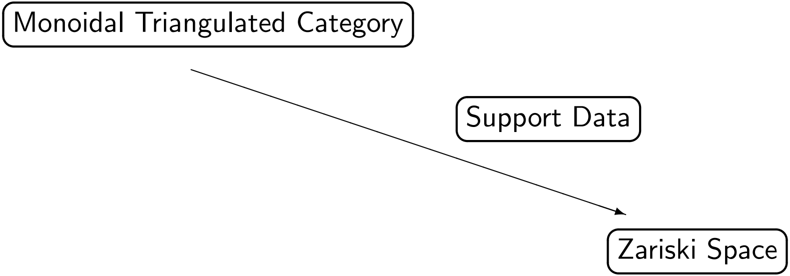
given by

$$\Phi_\sigma(\mathbf{I}) = \bigcup_{M \in \mathbf{I}} \sigma(M), \qquad \Theta(W) = \mathbf{I}_W,$$

where $\mathbf{I}_W = \{M \in \mathbf{K}^c \mid \sigma(M) \subseteq W\}$*. Moreover, there is a homeomorphism*

$$f : X \to \operatorname{Spc}(\mathbf{K}^c).$$

Recap



Monoidal Triangulated Category
Support Data
Zariski Space

Finite Group Schemes [Symmetric MΔC]

Example

Let G be a finite group (scheme), $A := \mathrm{H}^\bullet(G, k) = \mathrm{Ext}^\bullet_G(k, k)$ be the cohomology ring. Set $\mathbf{K}^c = \mathrm{stmod}(G)$ and $X = \mathrm{Proj}(\mathrm{Spec}(A))$.

(i) $\{$thick $\otimes$-ideals of $\mathbf{K}^c\}$ are in one-to-one correspondence with $\mathcal{X}_{sp}$.

(ii) $\mathrm{Spc}(\mathbf{K}^c) \cong \mathrm{Proj}(\mathrm{Spec}(A))$.

The (classifying) support data is given by

$$W(M) = \{P \in \mathrm{Proj}(\mathrm{Spec}(A)) : \mathrm{Ext}^\bullet_G(M, M)_P \neq 0\}.$$

Restricted Lie Algebras

Example

Let $G = \mathrm{GL}_n(k)$ be the group of invertible $n \times n$ matrices over k an algebraically closed field of characteristic $p > 0$. Let $\mathfrak{g} = \mathfrak{gl}_n(k)$, G_1 the first Frobenius kernel, and $\mathcal{N}$ be the nilpotent $n \times n$ matrices.

Theorem (Friedlander-Parshall, Andersen-Jantzen, 1984)

Let $p > h$

(a) $\mathrm{H}^{2\bullet}(G_1, k) \cong k[\mathcal{N}]$

(b) $\mathrm{H}^{2\bullet+1}(G_1, k) = 0$

Then the Balmer Spectrum has a concrete realization via nilpotent matrices.

$$\mathrm{Spc}(\mathrm{stmod}(G_1)) \cong \mathrm{Proj}(\mathrm{Spec}(k[\mathcal{N}])).$$

Perfect Complexes [Symmetric MΔC]

Example

Let R be a commutative Noetherian ring, $\mathbf{K}^c = D^b_{perf}(R)$ and $X = \text{Spec}(R)$. Then

(i) $\{$thick $\otimes$-ideals of $\mathbf{K}^c\}$ are in one-to-one correspondence with $\mathcal{X}_{sp}$.

(ii) $\text{Spc}(\mathbf{K}^c) \cong \text{Spec}(R)$.

The support data which gives this classification is

$$W(C_\bullet) = \{P \in \text{Spec}(R) : \mathrm{H}_*(C_\bullet)_P \neq 0\}.$$

Benson-Witherspoon Hopf Algebras [Non-symmetric MΔC]

Benson and Witherspoon considered the stable module categories of Hopf algebras of the form

$$A := (k[G]\#kH)^*,$$

where

- G and H are finite groups with H acting on G by group automorphisms,
- k is a field of positive characteristic dividing the order of G,
- kH is the group algebra of H, $k[G]$ is the dual of the group algebra of G,
- A is a non-cocommutative Hopf algebra.

Enlightening Example

Example

Let p be a prime number and n be a positive integer. Benson and Witherspoon analyzed the situation for $G := (\mathbb{Z}/p\mathbb{Z})^n$, $H := \mathbb{Z}/n\mathbb{Z}$ (with H cyclically permuting the factors of G) and k a field of characteristic p,

In this case, A admits a non-projective finite dimensional module M such that $M \otimes M$ is projective. In particular, if W is the cohomological support then

$$W(M \otimes M) \neq W(M) \cap W(M).$$

Classification and the Balmer Spectrum

Theorem (Nakano-Vashaw-Yakimov 19)

Let $A = (k[G]\# kH)^*$ where G and H are finite groups with H acting on G and k is a base field of positive characteristic dividing the order of G. Let $R = \mathrm{H}^\bullet(A, k)$ and $X = H\text{-}\mathrm{Proj}(R)$. The following hold:

(a) *There exists a bijection*

$$\{\textit{thick tensor ideals of } \mathrm{stmod}(A)\} \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \{\textit{specialization closed sets of } X\}$$

(b) *There exists a homeomorphism $f : H\text{-}\mathrm{Proj}(R) \to \mathrm{Spc}(\mathrm{stmod}(A))$.*

Tensor Product Question

Open Question: When does a support datum $\sigma : \mathbf{K} \to \mathcal{X}_{sp}(X)$ possesses the *tensor product property*

$$\sigma(M \otimes N) = \sigma(M) \cap \sigma(N), \quad \forall M, N \in \mathbf{K}?$$

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- For cohomological supports for modular representations of finite groups (Carlson, Avrunin-Scott) and for finite group schemes (Friedlander-Pevtsova), it is known to hold.
- Many people have been interested in this question for arbitrary finite-dimensional Hopf algebras.

Connection with Completely Prime Ideals

Theorem (NVY20)

For every monoidal triangulated category $\mathbf{K}$, *the following are equivalent:*

(a) *The universal support datum* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc} \mathbf{K})$ *has the tensor product property*

$$V(M \otimes N) = V(M) \cap V(N), \quad \forall M, N \in \mathbf{K}.$$

(b) *Every prime ideal of* $\mathbf{K}$ *is completely prime.*

Universal support datum: $V : \mathbf{K} \to \mathcal{X}$: $V(M) = \{\mathbf{P} \in \operatorname{Spc} \mathbf{K} : \ M \notin \mathbf{P}\}$.

Connections with Nilpotent Elements

Theorem (NVY20)

Let $\mathbf{K}$ *be a monoidal triangulated category in which every object is rigid. If* $\mathbf{K}$ *has a non-zero nilpotent object* M *(i.e.,* $M \ncong 0$ *but* $M^{\otimes n} := M \otimes \cdots \otimes M \cong 0$*, for some* $n > 0$*) then not all prime ideals of* $\mathbf{K}$ *are completely prime. As a consequence, the universal support datum* $V : \mathbf{K} \to \mathcal{X}(\mathrm{Spc}\,\mathbf{K})$ *does not have the tensor product property.*

Recall for the Benson-Witherspoon example, there exists a non-zero module M (not projective) such that $M \otimes M = (0)$ in $\mathbf{K}$. Therefore, the universal support datum does not satisfy the tensor product property. This implies that the cohomological (classifying) support datum does not satisfy the tensor product property.

A New and Bold Conjecture

Conjecture (Nakano-Vashaw-Yakimov 2022)

Let $\mathbf{T}$ *be finite tensor category and* $\underline{\mathbf{T}}$ *be its stable module category.*

(a) *There exists a homeomorphism*

$$\rho : \mathrm{Spc}\, \underline{\mathbf{T}} \to \mathrm{Proj}\, C^{\bullet}_{\underline{\mathbf{T}}}.$$

(b) *The monoids* $\mathrm{ThickId}(\underline{\mathbf{T}})$ *(thick tensor ideals) and* $\mathcal{X}_{sp}(\mathrm{Proj}\, C^{\bullet}_{\underline{\mathbf{T}}})$ *(specialization closed sets) are isomorphic.*

Here $C^{\bullet}_{\underline{\mathbf{T}}}$ is a new algebraic object called the *categorical center of the cohomology ring*.

The Categorical Center of the Cohomology Ring

The *categorical center* $C^{\bullet}_{\mathbf{T}}$ of the cohomology ring $R^{\bullet}_{\mathbf{T}}$ is the subalgebra generated by all homogeneous $g \in \mathrm{Hom}_{\mathbf{T}}(\mathbf{1}, \Sigma^i \mathbf{1})$, and for every simple object M the following diagram commutes::

$$\begin{array}{ccccc} \mathbf{1} \otimes M & \xrightarrow{\cong} & M & \xleftarrow[\cong]{} & M \otimes \mathbf{1} \\ \downarrow{\scriptstyle g \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}_M \otimes g} \\ \Sigma^i \mathbf{1} \otimes M & \xrightarrow{\cong} & \Sigma^i M & \xleftarrow[\cong]{} & M \otimes \Sigma^i \mathbf{1} \end{array}$$

Our Main Theorem [NVY22]

Theorem

Let $\mathbf{K}$ *be an* $M\Delta C$*, which is the compact part of a compactly generated* $M\Delta C$*,* $\widetilde{\mathbf{K}}$*. If*

(i) $\mathbf{K}$ *satisfies the (wfg) condition,*

(ii) $\operatorname{Proj} C^\bullet_{\mathbf{K}}$ *is a Zariski space and*

(iii) *the central cohomological support of* $\mathbf{K}$ *has an extension to a faithful extended weak support datum* $\widetilde{\mathbf{K}} \to \mathcal{X}(\operatorname{Proj} C^\bullet_{\mathbf{K}})$*,*

then the following hold:

(a) $\operatorname{Spc} \mathbf{K}$ *is homeomorphic to* $\operatorname{Proj} C^\bullet_{\mathbf{K}}$*.*

(b) *The map*

$$\Phi_{W_C} : \operatorname{ThickId}(\mathbf{K}) \to \mathcal{X}_{sp}(\operatorname{Proj} C^\bullet_{\mathbf{K}})$$

is an isomorphism of ordered monoids.

Thank you for your attention.