Homotopy QFT interpretation of “zesting” braided fusion categories

Colleen Delaney

Berkeley
UNIVERSITY OF CALIFORNIA

based on joint work with:
Cesar Galindo, Sung Kim,
Julia Plavnik, and Qing Zhang

Overview:

I. Motivation: physics and topology

II. Background: the braided zesting construction
  A. G-graded braided fusion categories
  B. Connections to 3D TQFT and topological order

III. The G-crossed braided zesting construction
  A. G-crossed braided fusion categories
  B. Connections to 3D Homotopy QFT and symmetry-enriched topological order

IV. Conclusions and Questions

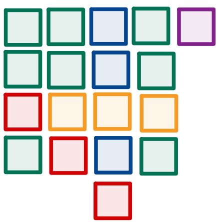
I. Motivation

Motivations for studying fusion categories

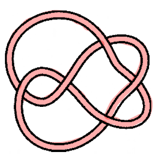
Modular fusion categories classify 3D Reshetikhin-Turaev TQFTs

(Certain symmetric monoidal functors Z: Cob → S)

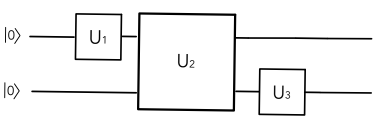
and give algebraic theories of 2d topological quantum matter



classification of modular categories/
topological phases of matter



their topological invariants/
observables



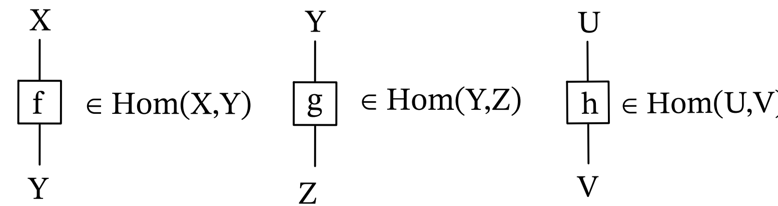
their representation theoretic invariants/
intrinsic quantum computational complexity

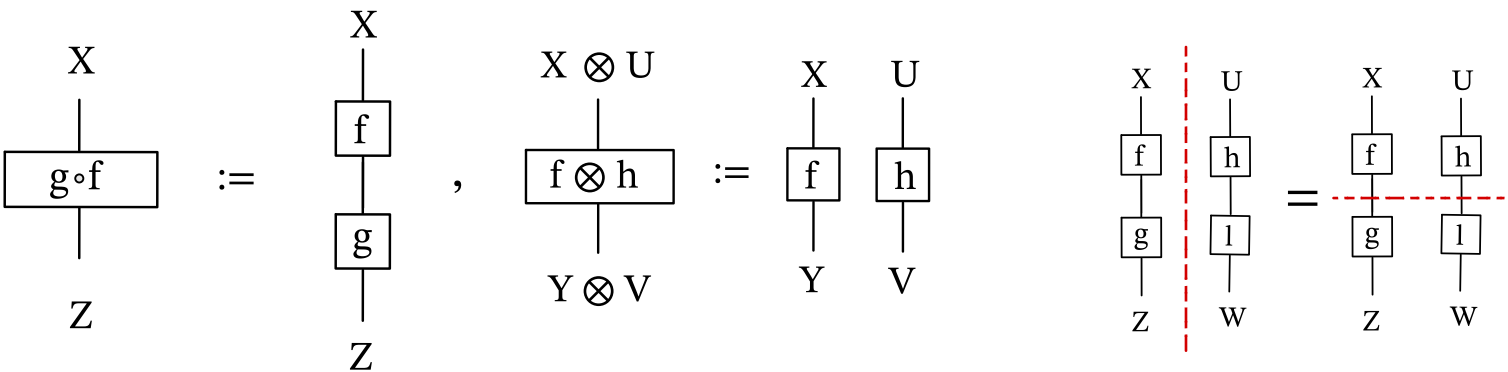
II. Background

Fusion categories

Recall that a fusion category is a *finite*, *semisimple* tensor category.
(I will write k for an algebraically closed field of characteristic 0 but mean $\mathbb{C}$)

Our main tool will be string diagrams:





"Ocneanu rigidity" : fusion categories admit no continuous deformations

G-graded fusion categories

Let G be a finite group.

A fusion category C is *G-graded* if

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \qquad \text{and} \qquad X_g \in \mathcal{C}_g,\ Y_h \in \mathcal{C}_h \Rightarrow X_g \otimes Y_h \in \mathcal{C}_{gh}$$

Grading is *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$

Every fusion category is faithfully graded by its *universal grading group*

Example: $\mathcal{C}(su(2),2)$ has simple objects $\mathbf{1}, \sigma, \psi$ with fusion rules

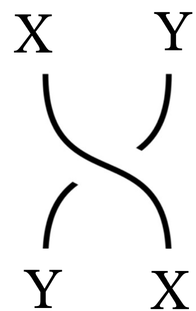
$$\sigma \otimes \psi = \psi \otimes \sigma = \sigma, \qquad \sigma \otimes \sigma = \mathbf{1} \otimes \psi, \qquad \psi \otimes \psi = \mathbf{1}$$

and is $\mathbb{Z}/2\mathbb{Z}$-graded with $\mathcal{C}(su(2),2) = \mathrm{sVec} \bigoplus \{\sigma\}$

Braided fusion categories

Denote the braiding isomorphisms by $\beta_{X,Y} : X \otimes Y \to Y \otimes X$ and depict by

$\beta_{X,Y} =$



<u>Note:</u> If a braided fusion category is G-graded then G is an abelian group because $X_g \otimes Y_h \cong Y_h \otimes X_g \Rightarrow \mathcal{C}_{gh} = \mathcal{C}_{hg}$ for all $g,h \in G$.

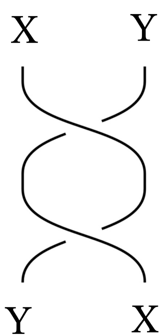
We will still use G to denote the abelian group.

Degeneracy of braiding

Rep(G), Rep(G,z)

Symmetric

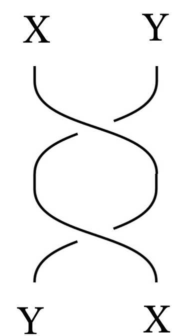
Non-degenerate



$= \mathrm{id}_{X \otimes Y}$

for all $X, Y \in \mathscr{B}$

if



$= \mathrm{id}_{X \otimes Y}$ for all $Y \in \mathscr{B}$

then $X \cong \mathbf{1}$

Idea of zesting

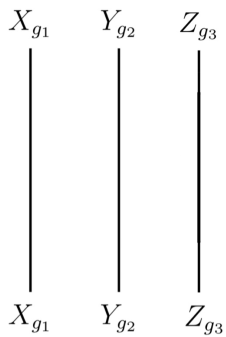
Main idea: modify fusion rules and ask if it categorifies with the desired structure (fusion, braided, ribbon)

$$X_g \otimes Y_h \longrightarrow X_g \overset{\lambda}{\otimes} Y_h = X_g \otimes Y_h \otimes \underbrace{\lambda(g,h)}_{\in \mathcal{C}_e \cap \mathcal{C}_{pt}}$$

Monoidal categorification of zested fusion rule $X_g \otimes Y_h \otimes \lambda(g,h)$

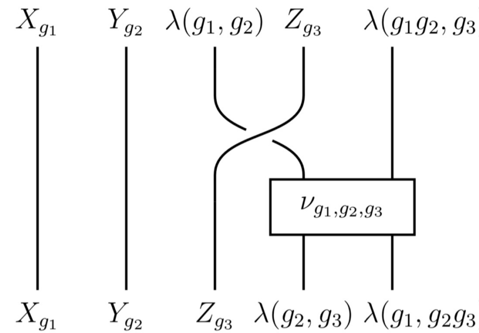
Before:

$\alpha_{X,Y,Z} =$

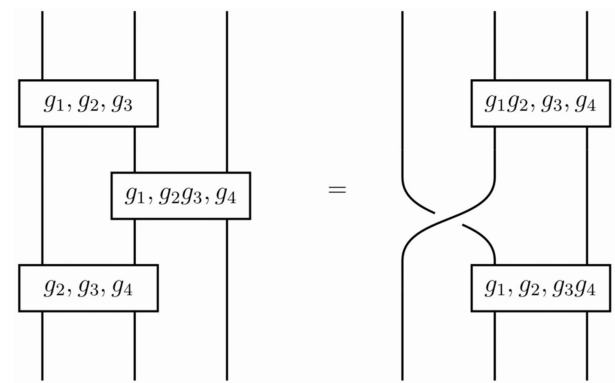


After:

$\alpha^{\lambda,\nu}_{X,Y,Z} =$



where

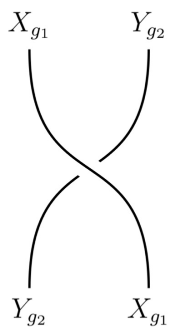


(sometimes I will supress the λ, ν to save space)

Braided categorification of zested fusion rule $X_g \otimes Y_h \otimes \lambda(g,h)$

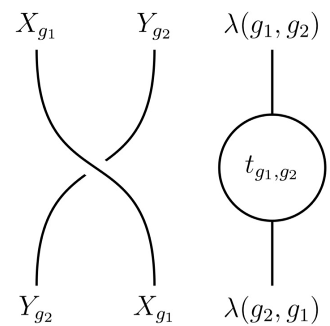
Before:

$\beta_{X,Y}$ =

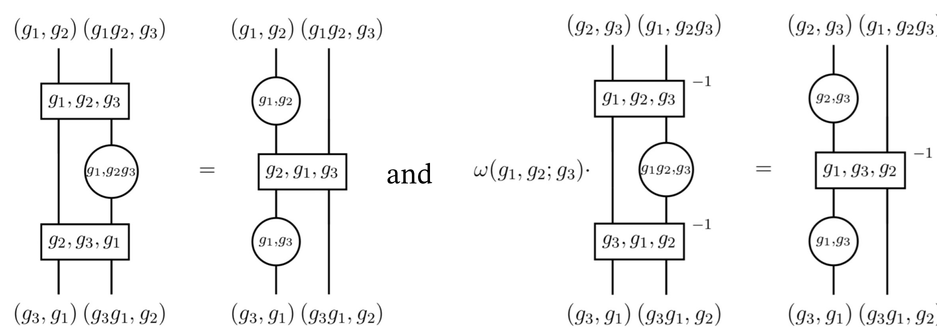


After:

$\beta^{\lambda,t}_{X,Y}$ =



where



The zesting construction

Let $\mathscr{B}$ be a braided fusion category with G-grading. WLOG can assume $\mathscr{B}$ strict.

"Associative zesting"

1. Pick 2-cocycle $\lambda \in Z^2(G, \mathrm{Inv}(\mathscr{B}_e))$, i.e.

   $\lambda : G \times G \to \mathrm{Inv}(\mathscr{B}_e)$ such that $\lambda(g,h) \otimes \lambda(gh,k) \cong \lambda(h,k) \otimes \lambda(g,hk)$

   $(g, h) \mapsto \lambda(g,h)$

2. Pick 3-cochain $\nu \in C^3(G, k^\times)$ s.t.

   $\nu(g,h,k)\, \nu(g,hk,l)\, \nu(h,k,l) = \beta_{\lambda(g,h),\lambda(k,l)}\, \nu(gh,k,l)\nu(g,h,kl)$

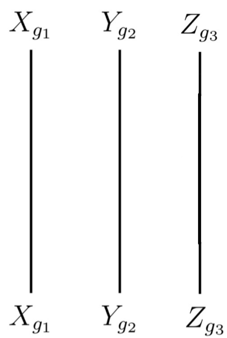
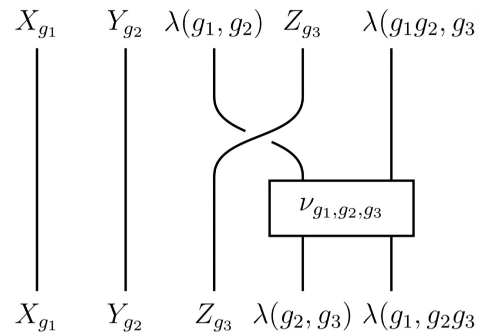
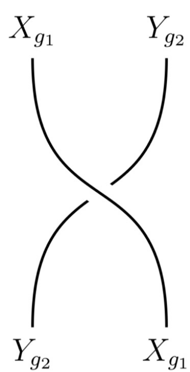
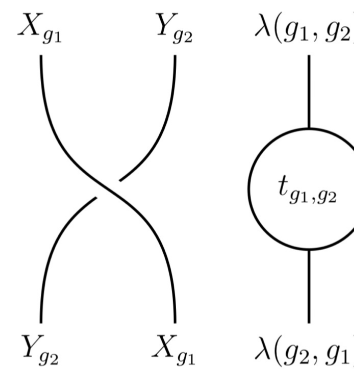
"Braided zesting"

3. Pick 2-cochain $t \in C^2(G,k^\times)$ s.t.

   $\nu(g,h,k)t(g,hk)\nu(h,k,g)=t(g,h)\nu(h,g,k)t(g,k)$

   $+$ a similar second equation

Summary of braided zesting construction

Before:	After:
$\alpha_{X,Y,Z} =$ 	$\alpha^{\lambda,\nu}_{X,Y,Z} =$ 
$\beta_{X,Y} =$ 	$\beta^{\lambda,t}_{X,Y} =$ 

Examples of zesting

- There are 8 (unitary) modular fusion categories with the same fusion rules as our first example $\mathcal{C}(su(2),2)$ and they are all related by zesting:

  Let $i, j, k \in \{0,1\}$

$$\lambda_a(i,j) = \begin{cases} \mathbf{1} & \text{if} \quad i+j < 2 \\ \psi & \text{if} \quad i+j \geq 2 \end{cases}$$

$$\nu_b(i,j,k) = \begin{cases} 1 & \text{if} \quad i+j < 2 \\ i^{k(a+2b)} & \text{if} \quad i+j \geq 2 \end{cases}$$

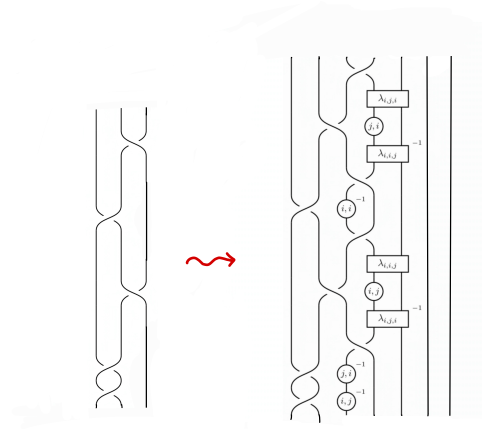
$$t_s(i,j) = s^{-ij} \quad \text{where } s = \pm\sqrt{i^{-(a+2b)}}$$

(here i means the imaginary #)

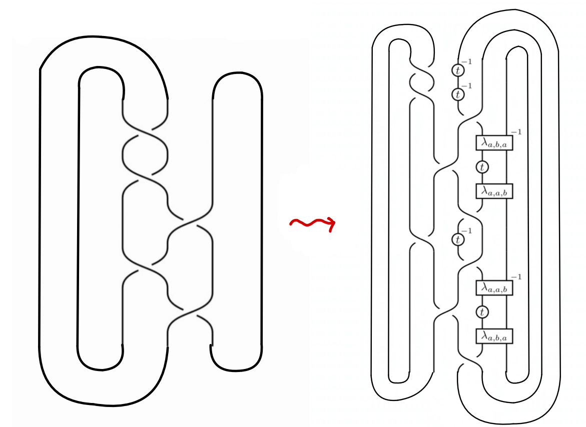
- Modular isotopes $\mathrm{Rep}(DG^{\omega})$ for $G = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ where p,q are certain odd primes

Properties of zesting

Braid group representations are (projectively) preserved.



Framed link invariants factorize, defining a new invariant of framed links colored by G that can be computed in polynomial time in the number of crossings.

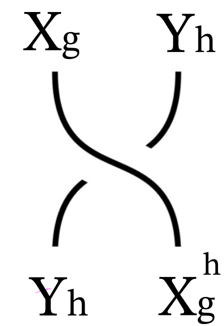


III. Theory of G-crossed braided“zesting”

G-crossed braided fusion categories

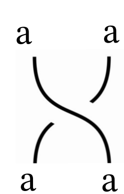
A fusion category C is *G-crossed braided* if it has

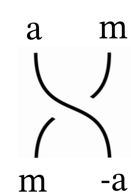
1. G-grading $\quad \mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$
2. G-action $\quad T: G \to \mathrm{Aut}(\mathcal{C})$
   $g \mapsto T_g \qquad$ s.t. $T_g(\mathcal{C}_h) \subset \mathcal{C}_{g^{-1}hg}$
3. G-braiding $\quad \beta_{X_g,Y_h}: X_g \otimes Y_h \to Y_h \otimes T_h(X_g)$

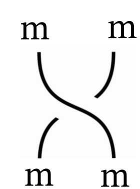


Example: Tambara-Yamagami fusion categories $\mathcal{C}$=TY(A, χ, τ)

$\mathbb{Z}/2\mathbb{Z}$- grading $\quad \{ a \mid a \in A \} \oplus \{ m \}$







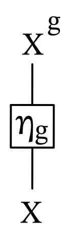
Relation between braided fusion categories with G-grading and G-crossed braided fusion categories

Example: Every G-graded braided fusion category is trivially G-crossed braided with G-action $T_g(Y_h) = Y_h$ and G-braiding $\beta_{X_g,Y_h} : X_g \otimes Y_h \cong Y_h \otimes X_g$

$\Rightarrow$ {braided fusion categories with G-grading} $\subset$ { G-crossed braided fusion categories}

Conversely, a G-crossed braided fusion category is braided if there exists a *trivialization* of the G-action functor T: G $\longrightarrow$ Aut(C), i.e. a monoidal natural isomorphism η of T with the identity functor on C

for all $g \in G$ have



natural isomorphisms satisfying conditions

Classification of G-crossed braided extensions of braided fusion categories

A braided fusion category $\mathscr{B}$ has a G-crossed braided extension if it admits a monoidal 2-functor $G \to \mathrm{BrPic}(\mathscr{B})$

These are classified by (ρ, λ, ω)

- group homomorphism $\rho : G \to \mathrm{Aut}(\mathscr{B})$
- $\lambda \in H^2_\rho(G, \mathrm{Inv}(\mathscr{B}))$
- $\omega \in H^3(G, k^\times)$

G-HQFTs and SET phases

G-crossed braided fusion categories with modular trivially-graded component classify *3D Homotopy QFTs with target BG*, which are expected to give the low-energy effective field theory description of symmetry defects in (bosonic) *symmetry-enriched topological phases of matter* in 2 spatial dimensions

The zesting construction

Let $\mathscr{B}$ be a braided fusion category with G-grading. WLOG can assume $\mathscr{B}$ strict.

"Associative zesting"

1. Pick 2-cocycle $\lambda \in Z^2(G, \mathrm{Inv}(\mathscr{B}_e))$, i.e.

   $\lambda : G \times G \to \mathrm{Inv}(\mathscr{B}_e)$ such that $\lambda(g,h) \otimes \lambda(gh,k) \cong \lambda(h,k) \otimes \lambda(g,hk)$

   $(g, h) \mapsto \lambda(g,h)$

2. Pick 3-cochain $\nu \in C^3(G, k^\times)$ s.t.

   $\nu(g,h,k)\, \nu(g,hk,l)\, \nu(h,k,l) = \beta_{\lambda(g,h),\lambda(k,l)}\, \nu(gh,k,l)\nu(g,h,kl)$

"Braided zesting"

3. Pick 2-cochain $t \in C^2(G,k^\times)$ s.t.

   $\nu(g,h,k)t(g,hk)\nu(h,k,g) = t(g,h)\nu(h,g,k)t(g,k)$

   $+$ a similar second equation

The G-crossed braided zesting construction

G-crossed braided fusion category

Let $\mathscr{B}$ be a ~~braided fusion category with G-grading~~. WLOG can assume $\mathscr{B}$ strict.

“*G-crossed* Associative zesting”

1. Pick 2-cocycle $\lambda \in Z^2_G(G, \mathrm{Inv}(\mathscr{B}_e))$ *with G-action*, i.e.

   $\lambda : G \times G \to \mathrm{Inv}(\mathscr{B}_e)$ such that $\lambda(g,h)^k \otimes \lambda(gh,k) \cong \lambda(h,k) \otimes \lambda(g,hk)$

   $(g, h) \mapsto \lambda(g,h)$

2. Pick 3-cochain $\nu \in C^3(G, k^\times)$ s.t.

   $\nu(g,h,k)^l\, \nu(g,hk,l)\, \nu(h,k,l) = \beta_{\lambda(g,h),\lambda(k,l)}\, \nu(gh,k,l)\nu(g,h,kl)$

“~~Braided zesting~~”

~~3. Pick 2-cochain $t \in C^2(G,k^\times)$ s.t.~~

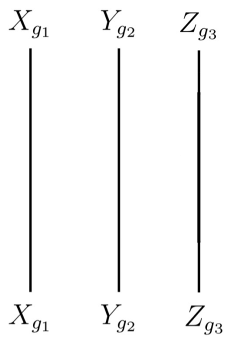
~~$\nu(g,h,k)t(g,hk)\nu(h,k,g)=t(g,h)\nu(h,g,k)t(g,k)$~~

~~+ a similar second equation~~

Monoidal categorification of zested fusion rule $X_g \otimes Y_h \otimes \lambda(g,h)$

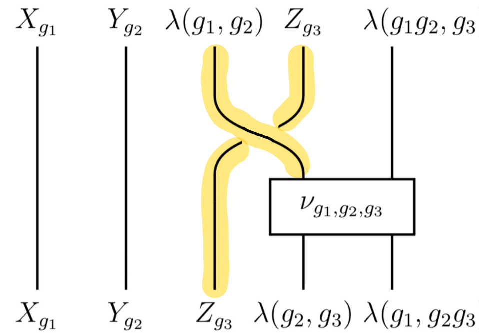
Before:

$\alpha_{X,Y,Z} =$

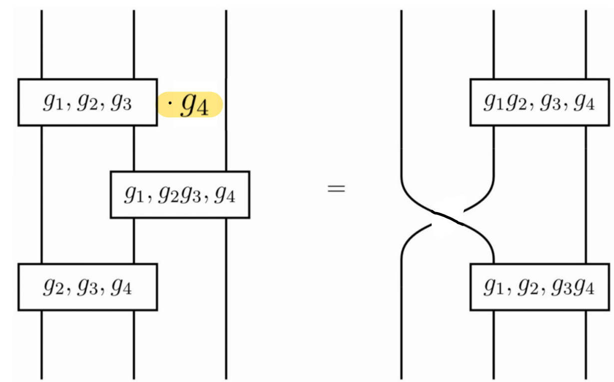


After:

$\alpha^{\lambda,\nu}_{X,Y,Z} =$



where



Structure morphisms of zested G-crossed braided fusion categories

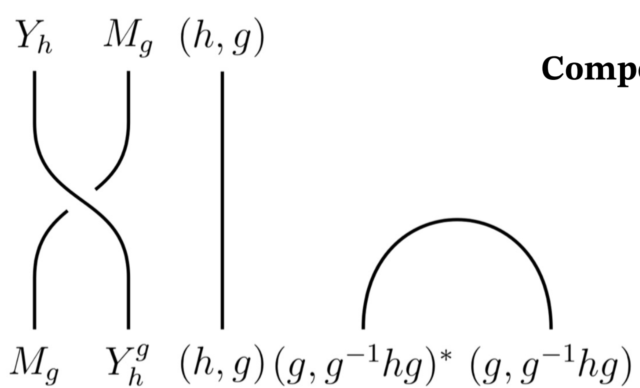
Theorem: the fusion category obtained from associative zesting of a G-crossed braided fusion category is automatically G-crossed braided with:

G-action on objects:

$$T_g^{(\lambda,\nu)} : \mathcal{C}_h^{(\lambda,\nu)} \to \mathcal{C}_{g^{-1}hg}^{(\lambda,\nu)}$$
$$Y_h \mapsto Y_h^g \otimes \lambda(h,g) \otimes \lambda(g, g^{-1}hg)^*$$

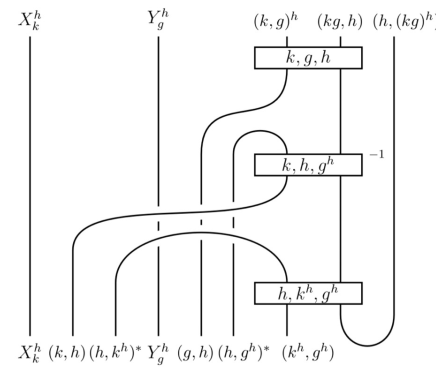
G-braiding:

$c_{Y_h,M_g}^{(\lambda,\nu)} =$



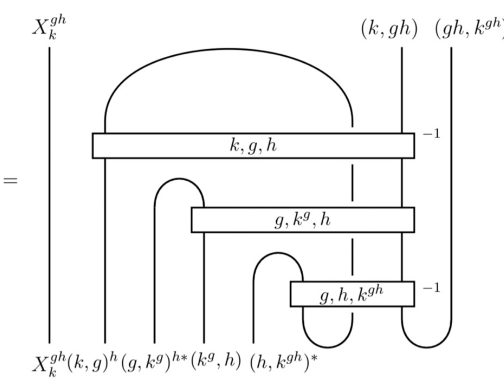
Tensorators:

$\mu_{h^\lambda}^{X,Y} =$



Compositors:

$\gamma_{g^\lambda,h^\lambda}^X =$



Relation between extension theory and G-crossed braided zesting

Theorem: Any two G-crossed extensions of a braided fusion category $\mathcal{B}$ with the same group homomorphism $\rho: G \to \mathrm{Pic}(\mathcal{B})$ are related by G-crossed braided zesting.

if $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g, \mathcal{C}_e = \mathcal{B}$ $\qquad$ $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g, \mathcal{D}_e = \mathcal{B}$

and $\mathcal{C}_g = \mathcal{D}_g$ as $\mathcal{B}$-module categories for all $g \in G$

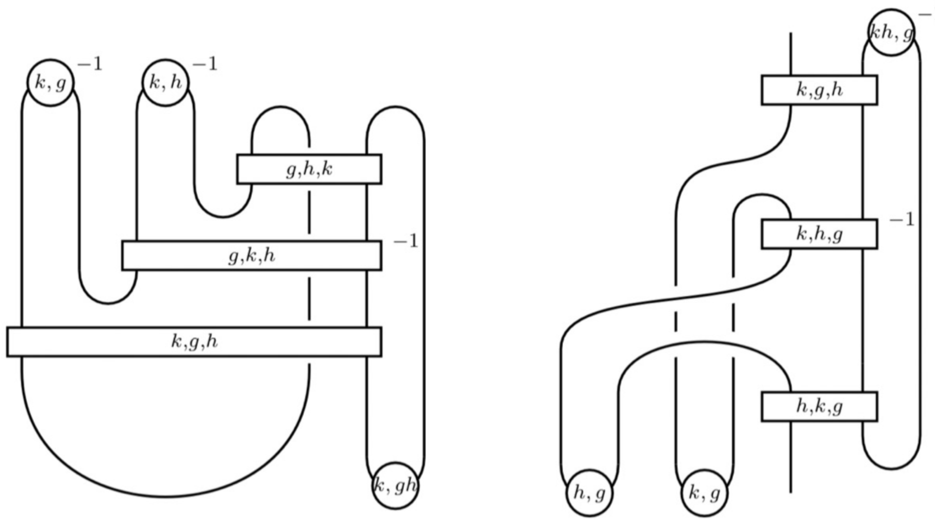
then there exists G-crossed zesting data (λ,ν) such that $\mathcal{D} \cong \mathcal{C}^{\lambda,\nu}$

Recovering braided zesting from G-crossed braided zesting*

Theorem:

Every zesting (λ,ν,t) of a braided fusion category $\mathcal{B}$ (where λ takes values in a symmetric subcategory of $\mathcal{B}_e$) comes from the G-crossed zesting (λ,ν) together with a trivialization η.

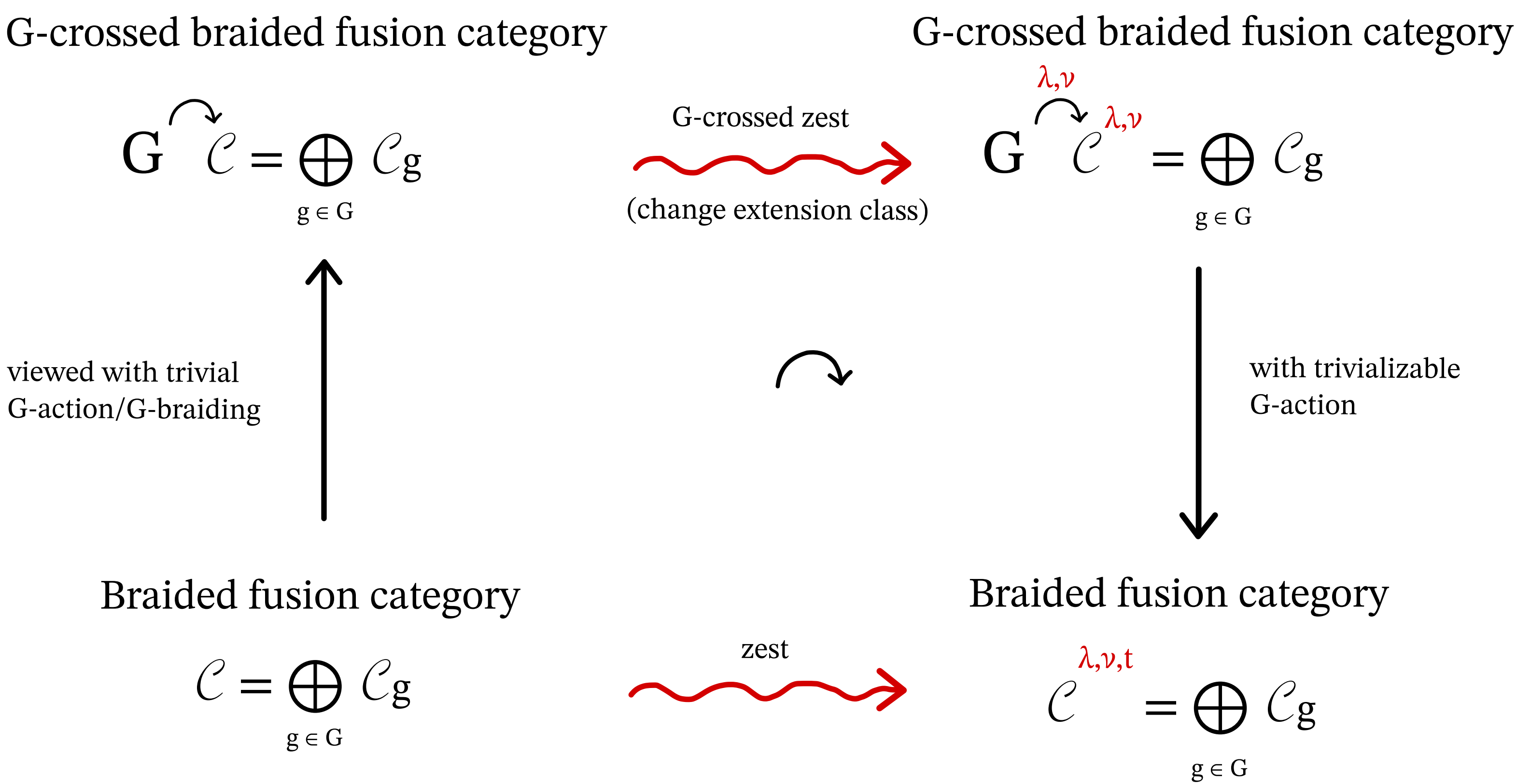
Proof:



(η satisfies the definition of a trivialization iff t satisfies the braided zesting equations)

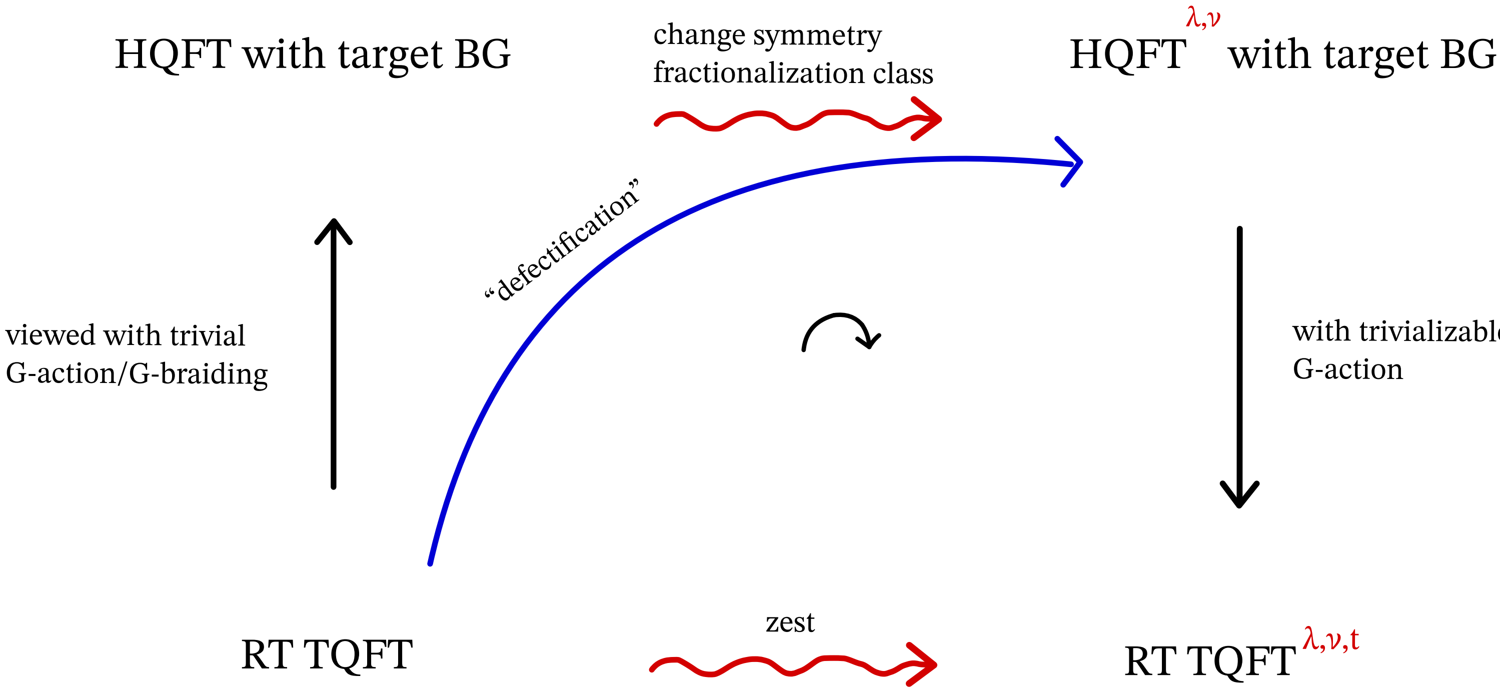
*technically this only works if λ takes values in a symmetric subcategory

Recovering braided zesting from G-crossed braided zesting*



*technically this only works if λ takes values in a symmetric subcategory

Physics interpretation



Connecting zesting to representation theory

What is the right notion of “zesting” (weak) Hopf algebras or vertex operator algebras so that the following diagram commutes?

$$
\begin{array}{ccc}
\mathrm{Rep}(H) & \xrightarrow{\text{zest}} & \mathrm{Rep}(H)^{\lambda,\nu,t} \\
\uparrow & & \uparrow \\
H & \xrightarrow[???]{} & H^{\lambda,\nu,t}
\end{array}
$$

And how much of this story still works when we relax our assumptions about finiteness and semisimplicity?

Thanks!