Newton–Okounkov bodies and minimal models for cluster varieties

Alfredo Nájera Chávez

joint with Lara Bossinger, Man-Wai Cheung and Timothy Magee

arXiv:2305.04903

Outline

- Gross-Hacking-Keel-Kontsevich (GHKK) generalized the polytope construction of a toric variety to the world of cluster varieties.

Goal: Reverse this construction and understand this process from the view-point of Newton-Okounkov bodies (NO bodies).

Applications:

- Describe Rietsch-Williams (RW) NO bodies for Grassmannians using tropical geometry and mirror symmetry as a type of *convex hulls*.
- Produce families of examples of NO bodies with the wall-crossing phenomenon in the sense of Escobar-Harada.

To define a *pair* of *mirror dual* cluster varieties of dimension r we need:

- A quiver Q without loops $\overset{\curvearrowright}{\bullet}$ nor 2-cycles $\bullet \rightleftarrows \bullet$.
- A subset $F \subset Q_0$ of frozen vertices.

$$(Q,F) \begin{array}{l} \nearrow \mathcal{A}_{(Q,F)} \\ \searrow \mathcal{X}_{(Q,F)} \end{array}$$

$$\text{If} \quad \mathcal{V} = \begin{cases} \mathcal{A}_{(Q,F)} \\ \mathcal{X}_{(Q,F)} \end{cases} \qquad \text{then} \qquad \mathcal{V}^{\vee} = \begin{cases} \mathcal{X}_{(Q,F)} \\ \mathcal{A}_{(Q,F)} \end{cases}$$

Rough description

Let

- $N \cong \mathbb{Z}^r$ and $M := \mathrm{Hom}(N, \mathbb{Z})$
- $T_N := N \otimes \mathbb{C}$ and $T_M := M \otimes \mathbb{C}$.

Then

$$\mathcal{A}_{(Q,F)} = \bigcup_{\mathbf{s} \in \Delta(Q,F)_0} T_{N;\mathbf{s}} \qquad \text{and} \qquad \mathcal{X}_{(Q,F)} = \bigcup_{\mathbf{s} \in \Delta(Q,F)_0} T_{M;\mathbf{s}},$$

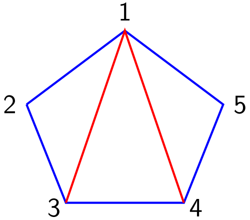
where

- $\Delta(Q, F)$ is a the *cluster complex* associated to (Q, F)
- $\Delta(Q, F)_0$ is the set of vertices of $\Delta(Q, F)$
- each torus has preferred coordinates: the cluster coordinates.
- the change of coordinates is given by cluster transformations.

Example

A triangulation **s** of a pentagon defines:

- a torus in the affine cone $C(\mathrm{Gr}_2(\mathbb{C}^5))$

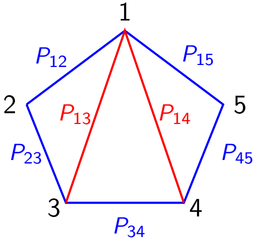


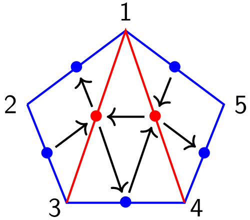
$T_{\mathbf{s}} = \{A : P_{ij}(A) \neq 0 \text{ for every arc } ij \text{ in } \mathbf{s}\}$

Example

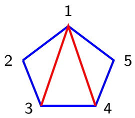
A triangulation **s** of a pentagon defines:

- a torus in the affine cone $C(\mathrm{Gr}_2(\mathbb{C}^5))$
- a quiver

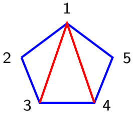


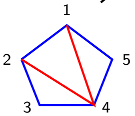


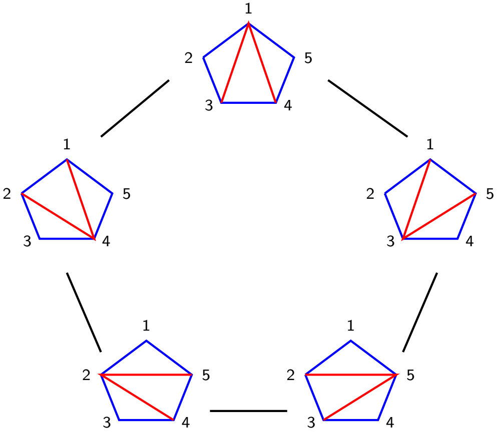
$$T_{\mathbf{s}} = \{A : P_{ij}(A) \neq 0 \text{ for every arc } ij \text{ in } \mathbf{s}\}$$

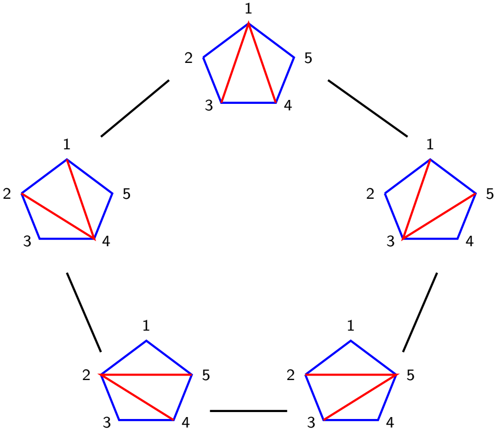

1
2
5
3
4

$$P_{24} = \frac{P_{23}P_{14} + P_{15}P_{12}}{P_{13}}$$






1
2
5
3
4
1
2
5
3
4
1
2
5
3
4
1
2
5
3
4
1
2
5
3
4



We obtain a cluster structure on $\mathrm{Gr}_2(\mathbb{C}^5) \setminus V(P_{12}P_{23}P_{34}P_{45}P_{15})$.

- $\Delta(Q, F)$ is a pentagon
- $\mathcal{A}_{(Q,F)} \overset{\text{codim } 2}{\cong} \mathrm{Gr}_2(\mathbb{C}^5) \setminus V(P_{12}P_{23}P_{34}P_{45}P_{15})$
- $\mathcal{A}_{(Q,F)} \cong \mathcal{X}_{(Q,F)}$

Minimal models

- By construction $\mathcal{A} = \mathcal{A}_{(Q,F)}$ and $\mathcal{X} = \mathcal{X}_{(Q,F)}$ are log-Calabi-Yau varieties with maximal boundary.
- In particular, they are smooth and have a canonical nowhere vanishing volume form Ω.

Definition

Let $\mathcal{V}$ be a cluster variety. An inclusion $\mathcal{V} \subset Y$ as an open subscheme of a normal variety Y is a partial minimal model (p.m.m.) of $\mathcal{V}$ if the canonical volume form on $\mathcal{V}$ has a simple pole along every irreducible divisor of Y contained in $Y \setminus \mathcal{V}$. It is a minimal model if Y is, in addition, projective. We call $Y \setminus \mathcal{V}$ the boundary of $\mathcal{V} \subset Y$.

Example

If $\overline{\mathcal{A}}$ is obtained by letting the frozen variables vanish then $\mathcal{A} \subset \overline{\mathcal{A}}$ is a partial minimal model.

Tropicalization

A cluster variety $\mathcal{V}$ has a well defined integral tropicalization

$$\mathcal{V}^{\text{trop}}(\mathbb{Z}) := \{\text{divisorial valuations } \nu : \mathbb{C}(\mathcal{V})^* \to \mathbb{Z} \mid \nu(\Omega) < 0\} \cup \{0\}$$

Divisorial valuation on $\mathcal{V}$: is a valuation of the form ord_D, where D is an irreducible divisor in a variety birational to $\mathcal{V}$.

Tropicalization

A cluster variety $\mathcal{V}$ has a well defined integral tropicalization

$$\mathcal{V}^{\text{trop}}(\mathbb{Z}) := \{\text{divisorial valuations } \nu : \mathbb{C}(\mathcal{V})^* \to \mathbb{Z} \mid \nu(\Omega) < 0\} \cup \{0\}$$

Divisorial valuation on $\mathcal{V}$: is a valuation of the form ord_D, where D is an irreducible divisor in a variety birational to $\mathcal{V}$.

$\rightsquigarrow$ can construct $\mathcal{V}^{\text{trop}}(\mathbb{R})$ by completion. So $\mathcal{V}^{\text{trop}}(\mathbb{Z}) \subset \mathcal{V}^{\text{trop}}(\mathbb{R})$.

Tropicalization

A cluster variety $\mathcal{V}$ has a well defined integral tropicalization

$$\mathcal{V}^{\text{trop}}(\mathbb{Z}) := \{\text{divisorial valuations } \nu : \mathbb{C}(\mathcal{V})^* \to \mathbb{Z} \mid \nu(\Omega) < 0\} \cup \{0\}$$

Divisorial valuation on $\mathcal{V}$: is a valuation of the form ord_D, where D is an irreducible divisor in a variety birational to $\mathcal{V}$.

$\rightsquigarrow$ can construct $\mathcal{V}^{\text{trop}}(\mathbb{R})$ by completion. So $\mathcal{V}^{\text{trop}}(\mathbb{Z}) \subset \mathcal{V}^{\text{trop}}(\mathbb{R})$.

Example

If $\mathcal{V} = T_L$ then $\mathcal{V}^{\text{trop}}(\mathbb{Z}) = L =$ cocharacter lattice of T_L.

Tropicalization

A cluster variety $\mathcal{V}$ has a well defined integral tropicalization

$$\mathcal{V}^{\text{trop}}(\mathbb{Z}) := \{\text{divisorial valuations } \nu : \mathbb{C}(\mathcal{V})^* \to \mathbb{Z} \mid \nu(\Omega) < 0\} \cup \{0\}$$

Divisorial valuation on $\mathcal{V}$: is a valuation of the form ord_D, where D is an irreducible divisor in a variety birational to $\mathcal{V}$.

$\rightsquigarrow$ can construct $\mathcal{V}^{\text{trop}}(\mathbb{R})$ by completion. So $\mathcal{V}^{\text{trop}}(\mathbb{Z}) \subset \mathcal{V}^{\text{trop}}(\mathbb{R})$.

Example

If $\mathcal{V} = T_L$ then $\mathcal{V}^{\text{trop}}(\mathbb{Z}) = L =$ cocharacter lattice of T_L.

Lemma

Every choice of cluster torus gives rise to identifications

$$\mathcal{X}^{\text{trop}}(\mathbb{Z}) \equiv M \qquad \mathcal{A}^{\text{trop}}(\mathbb{Z}) \equiv N$$

$$\mathcal{X}^{\text{trop}}(\mathbb{R}) \equiv M \otimes \mathbb{R} \qquad \mathcal{A}^{\text{trop}}(\mathbb{R}) \equiv N \otimes \mathbb{R}$$

If $\mathcal{V} = \bigcup T_{L;\mathbf{s}}$ we write $\mathcal{V}^{\text{trop}}_{\mathbf{s}}(\mathbb{Z})$ if we think of $\mathcal{V}^{\text{trop}}(\mathbb{Z})$ as L.

Conjecture (Fock-Goncharov 03')

The ring of regular functions $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$ has a canonical basis parametrized by $(\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z})$.

Facts

- The Fock-Goncharov (FG) conjecture is false in general.
- In 2014, GHKK introduced theta functions on cluster varieties and gave conditions ensuring that

$$\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) = \bigoplus_{q \in (\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z})} \mathbb{C} \cdot \vartheta_q^{\mathcal{V}}.$$

Conjecture (Fock-Goncharov 03')

The ring of regular functions $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$ has a canonical basis parametrized by $(\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z})$.

Facts

- The Fock-Goncharov (FG) conjecture is false in general.
- In 2014, GHKK introduced theta functions on cluster varieties and gave conditions ensuring that

$$\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) = \bigoplus_{q \in (\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z})} \mathbb{C} \cdot \vartheta_q^{\mathcal{V}}.$$

For most of our results we assume such decomposition exists.

This property has been verified in various families of examples.

Example

If $\mathcal{V} = T_L$ then $\mathcal{V}^\vee = T_{L^*}$ and $(\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z}) = L^*$. Then the canonical basis of $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$ is given by the characters of T_L.

Example $\mathrm{Gr}_2(\mathbb{C}^5)$

$$\left\{ \prod_{i \in \mathbb{Z}_5} p_{ii+1}^{c_i} \prod_{\overline{ij} \in \mathrm{int}(\mathbf{s})} p_{ij}^{a_{ij}} \mid \mathbf{s} \text{ is a triangulation, } a_{ij} \in \mathbb{Z}_{\geq 0},\ c_i \in \mathbb{Z} \right\}$$

is the set of theta functions on $\mathcal{A}_{(Q,F)}$.

Positive sets

Since we are assuming that $\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$ has a theta basis we can define the structure constants of this basis as follows:

$$\vartheta_p \vartheta_q = \sum_{r \in (\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z})} \alpha(p,q,r) \vartheta_r .$$

Definition

A closed subset $P \subseteq (\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{R})$ is *positive* iff

$$\forall a, b \in \mathbb{Z}_{\geq 0},\ \forall p \in aP(\mathbb{Z}),\ \forall q \in bP(\mathbb{Z})$$

$$\text{and } \forall r \in (\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z}) \quad \text{s.th. } \alpha(p,q,r) \neq 0$$

$$\text{then} \quad r \in (a+b)P.$$

Every positive set $P \subseteq (\mathcal{V}^\vee)^{\text{trop}}(\mathbb{R})$ determines a graded subring

$$R_P \subset \Gamma(\mathcal{V}, \mathcal{O}_\mathcal{V})[x].$$

Theorem (GHKK 14' + Keel-Yu 19')

Let $P \subset (\mathcal{V}^\vee)^{\text{trop}}(\mathbb{R})$ be a top dimensional, compact, rational positive polytope. Then we have an open inclusion

$$\mathcal{V} \hookrightarrow \text{proj}(R_P)$$

and a (multiparameter) toric degeneration

$$(\mathcal{V} \subset \text{proj}(R_p)) \rightsquigarrow (T_L \subset TV_P),$$

where TV_P is the toric variety associated to P. Moreover, $\text{proj}(R_p)$ is a minimal model for $\mathcal{V}$.

Aim:

1) Reveres this construction. Namely, for an open inclusion $\mathcal{V} \subset Y$ with Y projective and normal, construct a positive polytope $P_Y \subset (\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{R})$.
2) When this is possible show that P_Y is a Newton-Okounkov body.

NO bodies

Let Y be a d-dimensional variety (over $\mathbb{C}$). Suppose we have a valuation

$$\nu : \mathbb{C}(Y)^* \to (\mathbb{Z}^d, <).$$

Fix a line bundle

$$\mathcal{L} \to Y$$

and consider the section ring

$$R(\mathcal{L}) = \bigoplus_{k \geq 0} R_k(\mathcal{L}).$$

Definition

The Newton-Okounkov body associated to ν and $\mathcal{L}$ is

$$\Delta_\nu(\mathcal{L}) := \overline{\operatorname{conv}\left(\bigcup_{k \geq 1} \left\{ \frac{1}{k}\nu(f) \mid f \in R_k(\mathcal{L}) \setminus \{0\} \right\} \right)}.$$

- NO bodies are a far reaching generalization of both the Newton polytope of a Laurent polynomial and the polytope of a polarized projective toric variety.
- First introduced by Okounkov to develop the asymptotic theory of linear series of divisors.
- Its systematic study was developed by Lazarsfeld-Mustata 09' and Kaveh-Khovanskii 12'.
- Anderson 13' used them to construct (one parameter) toric degenerations of Y.

Cluster valuations

$$\widetilde{B}_{(Q,F)} = \begin{bmatrix} B^Q_{m\times m} \\ B^Q_{f\times m} \end{bmatrix}_{(m+f)\times(m)} = \left[\#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}\right]_{ij}$$

Theorem (Fujita-Oya 20')

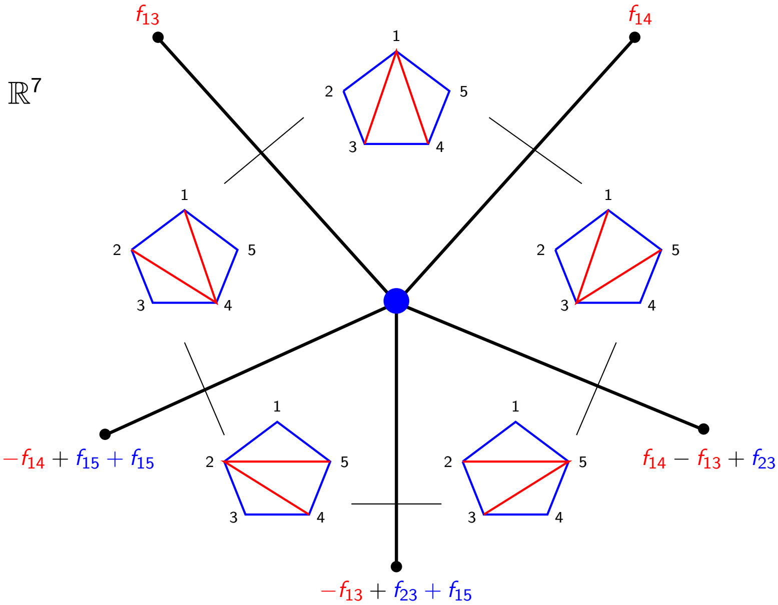
Suppose $\widetilde{B}_{(Q,F)}$ has full rank. Fix an identification $\mathcal{X}^{\text{trop}}_{\mathbf{s}}(\mathbb{Z}) \equiv M$. Then there exists a linear order $\prec_{\mathbf{s}}$ on M and a valuation

$$\mathbf{g}_{\mathbf{s}} : \mathbb{C}(\mathcal{A})^* \to (M, \prec_{\mathbf{s}})$$

such that $\mathbf{g}_{\mathbf{s}}(\vartheta_m) = -m$.

Remark (BCMNC 23')

If the FG conjecture holds for $\mathcal{X}$ then there is an analogous valuations on $\mathbb{C}(\mathcal{X})^*$.



● $= \text{cone}(f_{12}, f_{23}, f_{34}, f_{45}, f_{15})$

Assume $\mathcal{V}$ satisfies the FG conjecture. Fix a p.m.m. $\mathcal{V} \hookrightarrow Y$ and let $D_1, \ldots, D_n$ be the irreducible components of $Y \setminus \mathcal{V}$. Then

$$\mathrm{ord}_{D_i} \in \mathcal{V}^{\mathrm{trop}}(\mathbb{Z}) \qquad \text{and} \qquad \vartheta^{\mathcal{V}^\vee}_{\mathrm{ord}_{D_i}} \in \Gamma(\mathcal{V}^\vee, \mathcal{O}_{\mathcal{V}^\vee}).$$

Definition

The ϑ-superpotential associated to $\mathcal{V} \subset Y$ is

$$W_Y = \sum_{i=1}^{n} \vartheta^{\mathcal{V}^\vee}_{\mathrm{ord}_{D_i}}.$$

We say that $\mathcal{V} \subset Y$ has enough theta functions if

$$\{W_Y^{\mathrm{trop}} \geq 0\} := \{\mathrm{ord}_{D'} \in (\mathcal{V}^\vee)^{\mathrm{trop}}(\mathbb{Z}) \mid \mathrm{ord}_{D'}(W_Y) \geq 0\}$$

is a basis for $\Gamma(Y, \mathcal{O}_Y)$.

Remark

GHKK gave sufficient conditions ensuring that a p.m.m. $\mathcal{V} \subset Y$ has enough theta functions.

Example 1

Let $Y = \mathrm{Gr}_{n-k}(\mathbb{C}^n)$ and $\mathcal{X} = Y \setminus D$ be the positroid variety inside Y, where

$$D = \bigcup_{i=1}^{n} D_i \quad \text{and} \quad D_i = V(P_{[i,n-k+i-1]}).$$

Then

$$W_Y : \mathcal{A} \to \mathbb{C}$$

- $\mathcal{A}$ is the positrod variety inside $\mathrm{Gr}_k(\mathbb{C}^n)$
- $W_Y = \sum_{i=1}^{n} W_i$

$$W_i := \frac{P_{[i+1,i+k-1]\cup\{i+k+1\}}}{p_{[i+1,i+k]}}$$

W_Y is the superpotential introduced by Marsh-Rietsch.
Shen-Weng verified that this is the ϑ-superpotential.

Example 2

- $G = SL_n(\mathbb{C})$
- $Y = G/U$ is the base affine space
- $\mathcal{A} = G^{e,w_0} := B_- \cap Bw_0B$ is the double Bruhat cell.

Here B is a Borel subgroup, $B_- \subset G$ is the Borel subgroup opposite to B (i.e. $B \cap B^- =: T$ is a maximal torus) and w_0 the longest element in this Weyl group S_n is identified with a matrix representative in $N_G(T)/C_G(T)$.

Theorem (Magee 15')

The inclusion $\mathcal{A} \subset Y$ is a partial minimal model with enough theta functions and the Gelfand-Tsetlin cone is unimodular equivalent to the ϑ-superpotential cone $\{W_Y^{\mathrm{trop}} \geq 0\}$.

Quotients of $\mathcal{A}$

Every matrix of the form

$$\begin{bmatrix} B^Q_{m\times m} & -(B^Q_{f\times m})^{\mathrm{tr}} \\ B^Q_{f\times m} & *_{f\times f} \end{bmatrix}_{(m+f)\times(m+f)}$$

gives rise to a map cluster ensemble map

$$\mathcal{A}_Q \to \mathcal{X}_Q.$$

A choice of saturated sublattice $H \subset \mathrm{Ker}(p^*)$ gives rise to both an action and a fibration

$$T_H \curvearrowright \mathcal{A} \quad \text{and} \quad w_H : \mathcal{X} \to T_{H^*}.$$

We have a geometric quotient

$$(\mathcal{A}/T_H) = \bigcup_{\mathbf{s}\in\Delta(Q,F)_0} (T_N/T_H)_{\mathbf{s}}.$$

The weight map

We tropicalize w_H to obtain linear map

$$w_H^{\mathrm{trop}} : \mathcal{X}_{\mathbf{s}}^{\mathrm{trop}}(\mathbb{Z}) \to H^*.$$

Lemma (GHKK-BCMNC)

Every theta function on $\mathcal{A}$ is an eigenfunction with respect to the T_H action. Moreover, the set $(w_H^{\mathrm{trop}})^{-1}(h)$ parametrizes those theta functions on $\mathcal{A}$ of weight h.

Universal torsors

Definition

The Picard group of Y is the abelian group

$$\mathrm{Pic}(Y) = \{[\mathcal{L}] \mid \mathcal{L} \text{ is a line bundle over } Y\}.$$

The universal torsor of Y is

$$\mathrm{UT}_Y := \mathbf{Spec}_Y \left(\bigoplus_{[\mathcal{L}] \in \mathrm{Pic}(Y)} \mathcal{L} \right).$$

Assume $\mathrm{Pic}(Y)$ is free.

Then $\Gamma(\mathrm{UT}_Y, \mathcal{O}_{\mathrm{UT}_Y})$. is $\mathrm{Pic}(Y)$-graded $\rightsquigarrow$ action $T_{\mathrm{Pic}(Y)^*} \curvearrowright \mathrm{UT}_Y$.

Example

Let $G = SL_n(\mathbb{C})$. Consider the full flag variety G/B. Then

$$\mathrm{UT}_{G/B} \cong G/U.$$

Definition

Let $p : N \to M$ be a cluster ensemble map and $H \subset \mathrm{Ker}(p^*)$. We say that the pair (p^*, H) has the Picard property with respect to a partial minimal model $\mathcal{A} \subset \mathrm{UT}_Y$ if

- H and $\mathrm{Pic}(Y)^*$ have the same rank,
- the action of T_H on $\mathcal{A}$ coincides with the action of $T_{\mathrm{Pic}(Y)^*}$ on UT_Y restricted to $\mathcal{A}$.

Theorem (Bossinger-Cheung-Magee-NC)
Assume $\mathcal{A} \subset \mathrm{UT}_Y$ is a partial minimal model with enough theta functions and $\mathcal{A}$ has full rank. Suppose there is a pair (p^*, H) with the Picard property with respect to $\mathcal{A} \subset \mathrm{UT}_Y$.Then $\mathcal{A}/T_H \subset Y$ is a minimal model and for every $[\mathcal{L}] \in \mathrm{Pic}(Y)$ and every seed $\mathbf{s}$

$$\Delta_{\mathbf{g}_\mathbf{s}}(\mathcal{L}) = (w^{\mathrm{trop}})^{-1}(\mathcal{L}) \cap \{W^{\mathrm{trop}}_{\mathrm{UT}_Y} \geq 0\}.$$

In particular, $\Delta_{\mathbf{g}_\mathbf{s}}(\mathcal{L})$ is a positive set.

Concluding remarks

- Have a version of the theorem that makes no reference to universal torsors and considers Weil divisors as opposed to line bundles and minimal models for $\mathcal{V}$ of the form $\mathcal{A}$, $\mathcal{X}$, $\mathcal{A}/T_H$ or $\mathcal{X}_{\mathbf{1}}$.
- Can show that the NO bodies associated to Grassmannians by RW are of the form described in the previous theorem. This allows to describe every such NO body as the broken line convex hull of the valuations of the Plücker coordinates (after CMNC).
- Our examples produce various examples of NO bodies with the wall-crossing phenomena in the sense of Escobar-Harada.

References

And Dave Anderson, Okounkov bodies and toric degenerations. *Math. Ann.* **356** (2013), no. 3, 1183–1202.

BCMNC Lara Bossinger, Man-Wai Cheung, Timothy Magee and Alfredo Nájera Chávez, Newton–Okounkov bodies and minimal models for cluster varieties. arXiv:2305.04903 [math.AG]

CMNC Man-Wai Cheung, Timothy Magee and Alfredo Nájera Chávez, Compactifications of cluster varieties and convexity. *Int. Math. Res. Not. IMRN* 2022, no. 14, 1–54

EH Laura Escobar and Megumi Harada, Wall-crossing for Newton-Okounkov bodies and the tropical Grassmannian. *Int. Math. Res. Not. IMRN* (2022), no. 7, 5152–5203.

FG Vladimir Fock and Alexander Goncharov, Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. c. Norm. Supr.*, (4)**42** (2009), 865–930.

FO Naoki Fujita and Hironori Oya, Newton-Okounkov polytopes of Schubert varieties arising from cluster structures. arXiv:2002.09912 [math.RT]

GHKK Mark Gross, Paul Hacking, Sean Keel and Maxim Kontsevich, Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608 (2018)

KK Kiumars Kaveh and Askold Khovanskii, Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math. (2)*, **176** (2012), no. 2, 925–978.

LM Robert Lazarsfeld and Mircea Mustata, Convex bodies associated to linear series. *Ann. Sci. c. Norm. Supr.* (4) **42** (2009), no. 5, 783–835.

Mag Timothy Magee, Littlewood-Richardson coefficients via mirror symmetry for cluster varieties. *Proc. Lond. Math. Soc.* (3) **121** (2020), no. 3, 463–512.

MR Bethany R. Marsh and Konstanze Rietsch, The B-model connection and mirror symmetry for Grassmannians. *Adv. Math.* **366** (2020), 107027, 131 pp.

RW Konstanze Rietsch and Lauren Williams, Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians. *Duke Math. J.* **168** (2019), no. 18, 3437?3527.

SW Linhui Shen and Daping Weng, Cyclic sieving and cluster duality of Grassmannian. *SIGMA Symmetry Integrability Geom. Methods Appl.* **16** (2020), Paper No. 067, 41 pp.