Embeddings of Kac-Moody affine Grassmannian slices

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Introduction

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Main Theorem (rough statement)

There exist closed embeddings of affine Grassmannian slices

$$\overline{\mathcal{W}}^{\nu}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$$

for all symmetric Kac-Moody types

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Plan for this talk:

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2. Coulomb branches

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1. Affine Grassmannian slices
2. Coulomb branches
3. Closed embeddings and main theorem

Affine Grassmannian Slices

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- Dominance order:

$$\lambda \geq \mu \iff \lambda - \mu = \sum_i \mathbf{v}_i \alpha_i^\vee \text{ with all } \mathbf{v}_i \geq 0$$

Loop groups

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$$\mathrm{SL}_2(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a, b, c, d \in A, \ ad - bc = 1 \right\}$$

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- Congruence subgroup $G_1[[z^{-1}]] \subset G[[z^{-1}]]$ defined by

$$1 \longrightarrow G_1[[z^{-1}]] \longrightarrow G[[z^{-1}]] \xrightarrow{z^{-1} \mapsto 0} G \longrightarrow 1$$

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Call **affine Grassmannian slice** if μ is dominant.

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- *Mirković-Vybornov*: in type A, for $\lambda \geq \mu$ both dominant

$$\overline{\mathcal{W}}_\mu^\lambda \cong \mathcal{S}_{\tilde{\mu}} \cap \overline{\mathbb{O}_{\tilde{\lambda}}}$$

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because $\overline{G[z]z^\nu G[z]} \subseteq \overline{G[z]z^\lambda G[z]}$ iff $\nu \leq \lambda$

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- If $\mu \leq \lambda$ both dominant, then $\overline{\mathrm{Gr}^{\mu}} \subseteq \overline{\mathrm{Gr}^{\lambda}}$ and $\overline{\mathcal{W}}^{\lambda}_{\mu}$ provides transversal slice

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For any dominant coweight λ, action of Langlands dual $G^{\vee}$ on top Borel-Moore homology

$$\bigoplus_{\mu \leq \lambda} H^{BM}_{top}\left((\overline{\mathcal{W}}^{\lambda}_{\mu})^{+}\right) \;\cong\; \bigoplus_{\mu \leq \lambda} V(\lambda)_{\mu} \;=\; V(\lambda)$$

isomorphic to irrep $V(\lambda)$ with highest (co)weight λ

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- Usual (stronger) statement:

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called the **Coulomb branch** associated to (G, N)
- Arises as moduli space in quantum field theory, from $3d$ $\mathcal{N} = 4$ QFT associated to $(G, N \oplus N^*)$

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- More precisely, BFN define:

$$\mathcal{R}_{G,N} = \left\{ [g,n] \in \big(G((z)) \times N[[z]]\big)/G[[z]] \ : \ gn \in N[[z]] \right\}$$

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- They show $\mathcal{M}_C(G,N)$ is irreducible, normal, and Poisson

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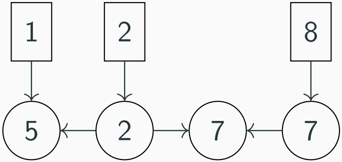

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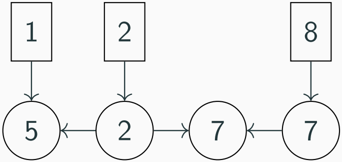

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- Define:

$$G = \prod_i \mathrm{GL}(\mathbf{v}_i),$$

$$N = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_j}) \oplus \bigoplus_i \mathrm{Hom}(\mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{v}_i})$$

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Theorem (Braverman-Finkelberg-Nakajima)

If quiver is oriented Dynkin diagram of finite ADE type, then

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- Independent of quiver orientation

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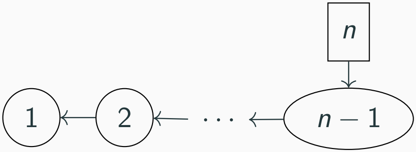

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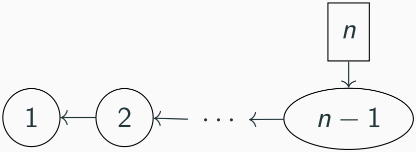
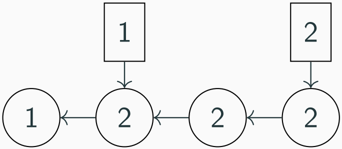


- $\overline{\mathbb{O}_{(3,1,1)}} \cong \overline{\mathcal{W}}_0^{(2,0,0,-1,-1)}$ corresponds to



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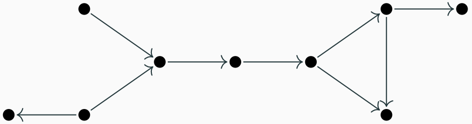

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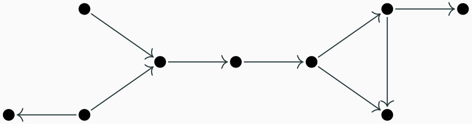

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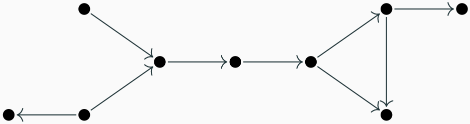

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Question

Does this definition satisfy expected properties?

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- In general, no concrete geometric model for $\overline{\mathcal{W}}_\mu^\lambda = \mathcal{M}_C(G, N)$

Closed embeddings

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- Explanation: Expect $\overline{\mathcal{W}}^{\lambda}_{\nu}$ transversal slice to $\overline{\mathcal{W}}^{\nu}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$, which should be conical by Kaledin

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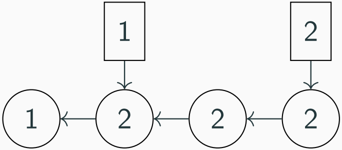

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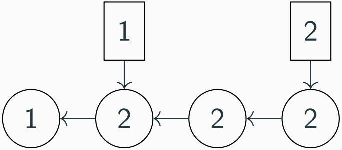

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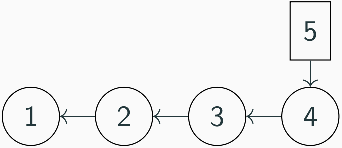

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Theorem (W.)

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- FMOs are explicit rational functions, in certain birational coordinates:

$$\sum_{\Gamma=(\Gamma_i)_{i\in I}} \frac{\prod_{i\to j}\prod_{r\in\Gamma_i, s\notin\Gamma_j}(w_{j,s}-w_{i,r})}{\prod_{r\in\Gamma_i, s\notin\Gamma_i}(w_{i,r}-w_{i,s})} \prod_{r\in\Gamma_i} \mathsf{u}_{i,r}$$

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- We show that in finite type, FMOs restrict to FMOs under

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- In all types, we show that this ansatz defines a closed embedding (under the assumptions of the theorem)

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$$\mathcal{A}_\hbar = H_*^{G[[z]] \rtimes \mathbb{C}^\times}(\mathcal{R}_{G,N})$$

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- In general the embedding $\overline{\mathcal{W}}^\nu_\mu \subseteq \overline{\mathcal{W}}^\lambda_\mu$ does not quantize, without fine tuning some parameters involved
$\implies$ geometric $H_*^{G[[z]]}(\mathcal{R}_{G,N}) \twoheadrightarrow H_*^{G'[[z]]}(\mathcal{R}_{G',N'})$ should be subtle

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- *Motto*: FMOs relevant to study of $\overline{\mathcal{W}}^\lambda_\mu$, even in finite types!

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I refuse to answer that question on the grounds that I don't know the answer.

- Douglas Adams