# Information complexity of convex optimization with integer variables

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Optimization and learning: theory and applications
Centre de Recherches Mathematiques (CRM)
Montreal, May 2025

# Convex optimization with integer variables

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\begin{aligned}
\text{minimize} \quad & f(x, y) \\
\text{subject to} \quad & (x, y) \in C, \\
& \underline{x \in \mathbb{Z}^n}, \; y \in \mathbb{R}^d
\end{aligned}
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$f : \underline{\mathbb{R}^n \times \mathbb{R}^d} \to \mathbb{R}$ convex

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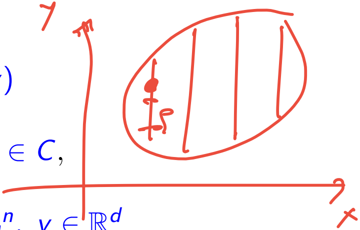


$C \subseteq \mathbb{R}^n \times \mathbb{R}^d$ closed, convex.
Contained in $\ell_\infty$ ball of radius $R > 0$.
Contains $\ell_\infty$ ball of radius $\rho > 0$ in the optimal fiber.

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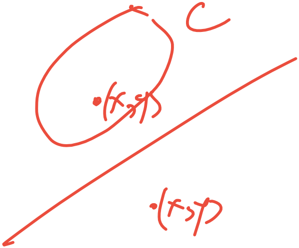


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First order oracle: separation for $C$, value + subgradient for $f$

# Information (a.k.a oracle) complexity

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$\epsilon > 0$

$(\hat{x}, \hat{y}) \in C$ s.t. $f(\hat{x}, \hat{y}) \leq f(x, y) + \epsilon$

$\forall (x, y) \in C$

Notion of $\epsilon$-approximate solutions

Information complexity: Minimum number of queries needed to (provably) report $\epsilon$-approximate solutions.

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$$\text{Information complexity} \quad \leq \quad \text{Algorithmic complexity}$$

# Complexity of convex optimization with integer variables

Information complexity (standard first-order oracle):

Lower bounds

$$d \geq 1: \quad \Omega\left(d2^n \log\left(\frac{MR}{\rho\epsilon}\right)\right)$$

$$d = 0: \quad \Omega\left(2^n \log(R)\right)$$

Upper bounds

$$n, d \geq 1: \quad O\left(d(n+d)2^n \log\left(\frac{MR}{\rho\epsilon}\right)\right)$$

$$d = 0: \quad O\left(n2^n \log(R)\right)$$

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Nemirovskii-Yudin 1970s
B.-Oertel 2017
B. 2023

$d = 0: \quad \Omega\left(2^n \log(R)\right)$

B.-Oertel 2017
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Upper bounds

$2^{O(n)} \log R$

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$n = 0: \quad O\left(d \log\left(\frac{MR}{\rho\epsilon}\right)\right)$ Nemirovskii-Yudin 1970s

# Complexity of convex optimization with integer variables

Overall (deterministic) complexity:

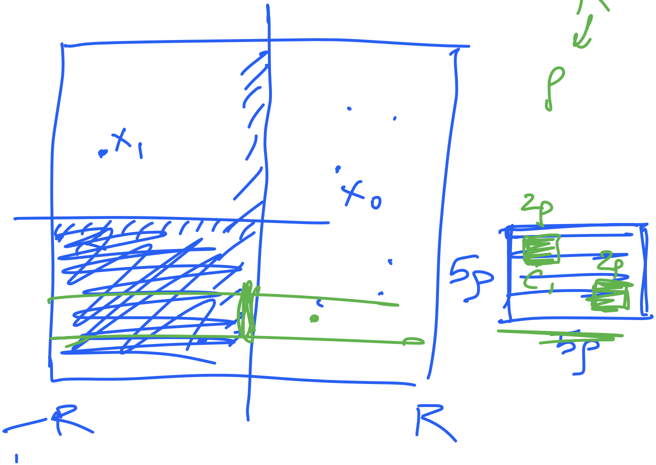
$$(n+d)^{O(n)} \mathrm{poly}\left(\log\left(\frac{MR}{\rho\epsilon}\right)\right)$$

$$\underbrace{n^{O(n)}}_{2^{n \log n}} \mathrm{poly}(d)\ \mathrm{poly}(\log \ldots)$$

# Information complexity lower bounds

$d \log \frac{nR}{\rho\epsilon}$ $\quad 2^n d \log \frac{R}{\rho}$ $\quad \min_{\boxed{x \in C}} f(x)$

$R \downarrow \rho$



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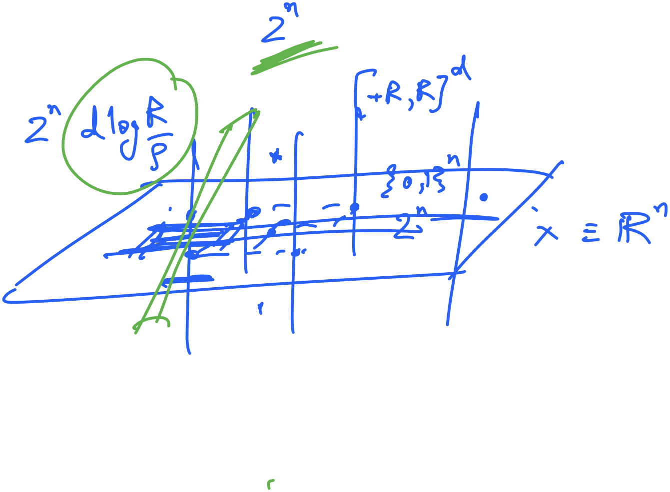
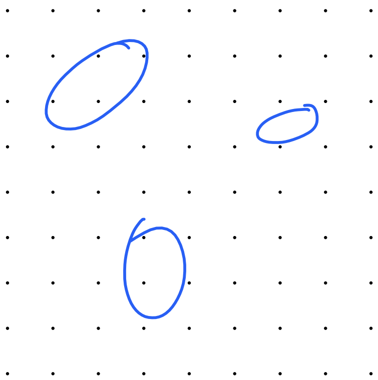


# Information complexity upper bounds

Feasibility problem with separation oracle for $C$



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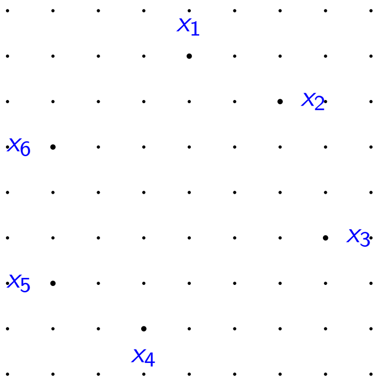
Assume we have queried at $x_1, x_2, \ldots, x_N$.



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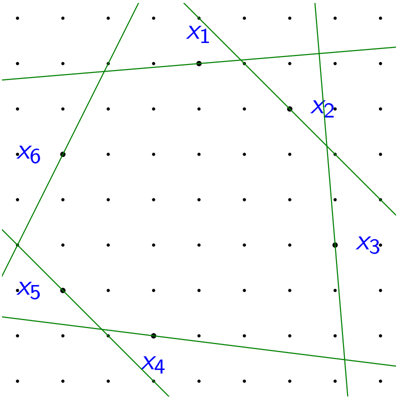

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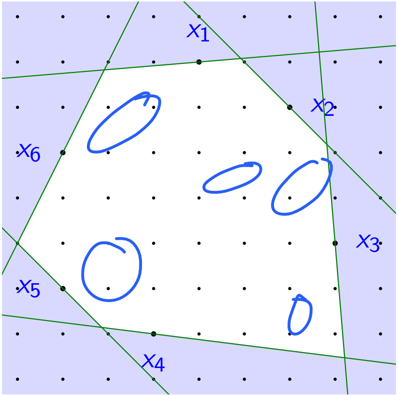

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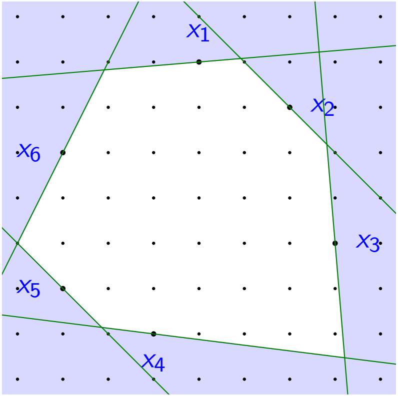
How should we choose $x_{N+1}$?



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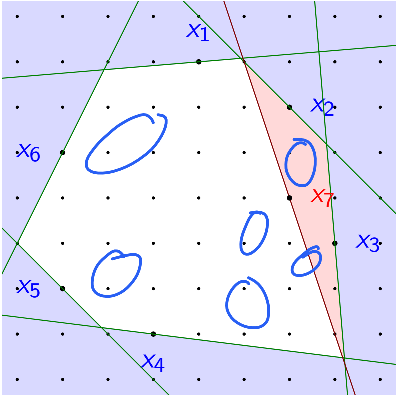

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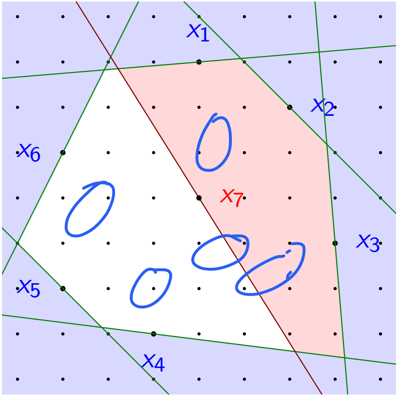

# Helly numbers and centerpoints

THEOREM Helly 1923, Doignon-Bell-Scarf 1970s, Averkov-Weismantel 2012

$C_1, \ldots, C_k \subseteq \mathbb{R}^n \times \mathbb{R}^d$ convex. If

$$C_1 \cap C_2 \cap \ldots C_k \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \emptyset,$$

then $\exists\ i_1, \ldots, i_{2^n(d+1)}$ such that

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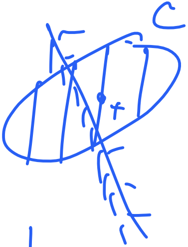


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$$1 - \frac{1}{2^n(d+1)}$$

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THEOREM Grünbaum 1960, Oertel 2013, B.-Oertel 2017
Probability measure $\mu$ supported on $\mathbb{Z}^n \times \mathbb{R}^d$. There exists $\underbrace{z^\mu \in \mathbb{Z}^n \times \mathbb{R}^d}_{\text{centerpoint}}$ such that for all halfspaces $H$ containing $z^\mu$,

$$\mu(H) \geq \frac{1}{2^n(d+1)}.$$

# Centerpoint based algorithm

- Let $P_0 := [-R, R]^{n+d}$.
- For $i = 1 \text{ to } N$
  - Compute centerpoint $z^i$ of $P_{i-1}$.
  - If $z^i \in C$, then return $z^i$ and STOP.
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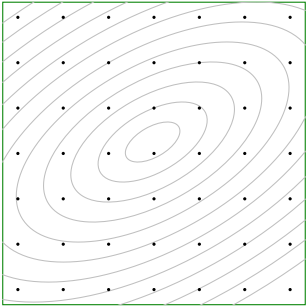

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$$\frac{1}{2^n(d+1)} = \frac{1}{2^n}$$

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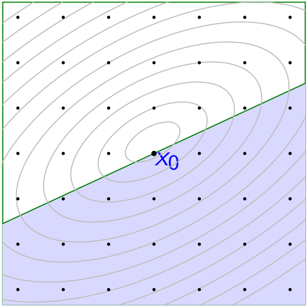
$$\left(1 - \frac{1}{2^n}\right)$$



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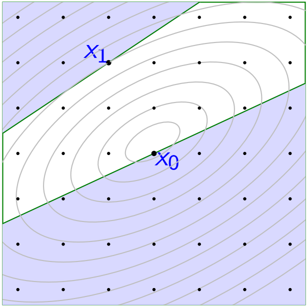

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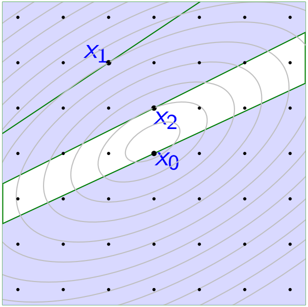

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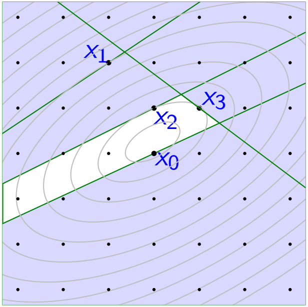

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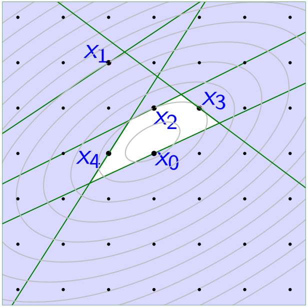
$< 1$

$R^n$

$\left(1 - \frac{1}{2^n}\right)$



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$\log R^n$

$\left(1-\frac{1}{2^n(d+1)}\right)$

$\cancel{\leq} 1$

$\approx \frac{1}{2}$

Upper bounds

$$n, d \geq 1: \quad O\left(d(n+d)2^n \log\left(\frac{MR}{\rho\epsilon}\right)\right)$$

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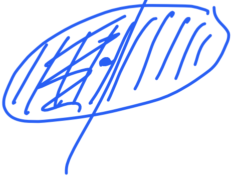

# What about other oracles?

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# What about other oracles?

Family of all possible binary queries: $\Theta\left((n+d)\log\left(\frac{MR}{\rho\epsilon}\right)\right)$ information complexity.

# What about other oracles?

Definition: Oracle based on first-order information:

1. First-order chart: Maps from instances to seperating hyperplanes, function values and (sub)gradients
2. Permissible queries: Maps from vectors/real numbers to a response set.

A query is now a pair $(z, h)$, where $z \in \mathbb{R}^n \times \mathbb{R}^d$ and $h$ is a permissible query.

# What about other oracles?

Examples:

1. Full-information first-order oracle: Permissible queries are identity functions.
2. Bit oracles: Permissible queries ask for a bit of the function value/some specific coordinate of (sub)gradient or separating hyperplane.
3. Directional oracles: Permissible queries ask for a sign of inner product between a specified direction and (sub)gradient/separating hyperplane/function value.
4. General binary oracles

# Information complexity with other oracles

A "transfer theorem" (B.-Kerger-Jiang-Molinaro 2024):

For <u>any</u> oracle using first-order information:

$\ell(d, R, \rho, M, \epsilon)$ lower bound for continuous convex optimization

$$\Downarrow$$

Mixed-integer problem has information complexity at least

$$\Omega\Big(2^n \ell(d, R, \rho, M, \epsilon)\Big)$$

# Information complexity with other oracles

Memory constrained convex optimization (Marsden, Sharan, Sidford, Valiant 2022, Blanchard, Zhang, Jaillet 2023):

$\ell(d, R, \rho, M, \epsilon) = \max\left\{d^{5/4}, d\log\left(\frac{MR}{\rho\epsilon}\right)\right\}$ for general binary oracle

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THEOREM (B.-Kerger-Jiang-Molinaro 2024)

$$\Omega\left(2^n \max\left\{d^{5/4}, d \log\left(\frac{MR}{\rho\epsilon}\right)\right\}\right)$$

mixed-integer lower bound for general binary oracle.

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# Some open questions

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Conjecture on mixed-integer centerpoints: improve bound from $\frac{1}{2^n(d+1)}$ to $\frac{1}{2^n e}$ .

Unify upper bounds, remove gap between lower and upper bounds

# Some open questions

Conjectured lower bound for general binary queries based on first order information (continuous convex optimization):

$$\Omega\left(d^2 \log\left(\frac{MR}{\rho\epsilon}\right)\right)$$

# Sample complexity of stochastic convex opt

$F : (\mathbb{R}^n \times \mathbb{R}^d) \times \mathcal{X} \to \mathbb{R}$

$C \subseteq \mathbb{R}^n \times \mathbb{R}^d$

$\xi$ is a random variable taking values in $\mathcal{X}$

$$\min_{z \in C \cap (\mathbb{Z}^n \times \mathbb{R}^d)} \quad \mathbb{E}_\xi[\ F(z, \xi)\ ]$$

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Solve the problem only given access to $n$ i.i.d. samples of $\xi$.

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Natural idea: Given samples $\xi^1, \ldots, \xi^n \in \mathcal{X}$, solve the deterministic problem

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Stochastic optimizers call this sample average approximation (SAA); machine learners call this empirical risk minimization (ERM).

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Given access to $n$ i.i.d. samples $\xi^1, \ldots, \xi^n \in \mathcal{X}$.

Uniform convergence:

$$\left| \frac{1}{n} \sum_{i=1}^{n} F(z,\xi^i) - \mathbb{E}_\xi[\, F(z,\xi) \,] \right| \leq \epsilon$$

for all $z \in [-R,R]^{n+d} \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ with prob. $1-\delta$

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for all $z \in [-R, R]^{n+d} \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ with prob. $1 - \delta$

SAA/ERM:

$$\mathbb{E}_\xi[\, F(z_{\mathsf{SAA}}, \xi)\, ] - \mathbb{E}_\xi[\, F(z^\star, \xi)\, ] \leq \epsilon$$

with prob. $1 - \delta$

# Sample complexity of stochastic convex opt

$$\min_{z \in C \cap (\mathbb{Z}^n \times \mathbb{R}^d)} \quad \mathbb{E}_\xi[\, F(z, \xi) \,]$$

Given access to $n$ i.i.d. samples $\xi^1, \ldots, \xi^n \in \mathcal{X}$.

Uniform convergence:

$$\left| \frac{1}{n} \sum_{i=1}^{n} F(z, \xi^i) - \mathbb{E}_\xi[\, F(z, \xi) \,] \right| \leq \epsilon$$

for all $z \in [-R, R]^{n+d} \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ with prob. $1 - \delta$

SAA/ERM:

$$\mathbb{E}_\xi[\, F(z_{\mathsf{SAA}}, \xi) \,] - \mathbb{E}_\xi[\, F(z^\star, \xi) \,] \leq \epsilon$$

with prob. $1 - \delta$

General learning: Report $\hat{z}(\xi^1, \ldots, \xi^n)$ such that

$$\mathbb{E}_\xi[\, F(\hat{z}, \xi) \,] - \mathbb{E}_\xi[\, F(z^\star, \xi) \,] \leq \epsilon$$

with prob. $1 - \delta$

# Continuous case ($n = 0$)

| | Lower bound | Upper bound |
|---|---|---|
| UC | $\Omega\left(d\frac{M^2R^2}{\epsilon^2} + \frac{M^2R^2}{\epsilon^2}\log\frac{1}{\delta}\right)$ | $O\left(d\frac{M^2R^2}{\epsilon^2}\log\left(\frac{MR}{\epsilon}\right) + \frac{M^2R^2}{\epsilon^2}\log\frac{1}{\delta}\right)$ |
| ERM | $\Omega\left(d\frac{MR}{\epsilon} + \frac{M^2R^2}{\epsilon^2}\log\frac{1}{\delta}\right)$ | $O\left(d\frac{MR}{\epsilon} + \frac{M^2R^2}{\epsilon^2}\log\frac{1}{\delta}\right)$ |
| GL | $\Omega\left(\frac{M^2R^2}{\epsilon^2}\log\frac{1}{\delta}\right)$ | $O\left(\frac{M^2R^2}{\epsilon^2}\log\frac{1}{\delta}\right)$ |

# With integer variables

| | Lower bound | Upper bound |
|---|---|---|
| UC | ?? | $O\left(\frac{M^2R^2}{\epsilon^2}\left(\begin{array}{c} n\log(R) + d\log\frac{MR}{\epsilon} \\ +\log\frac{1}{\delta}\end{array}\right)\right)$ |
| ERM | ?? | ?? |
| GL | $\Omega\left(\frac{M^2R^2}{\epsilon^2}\left(\begin{array}{c} n + d \\ +\log\frac{1}{\delta}\end{array}\right)\right)$ | ?? |

# Accompanying papers

https://arxiv.org/abs/2110.06172
(Math. Prog., Series B 2023)

https://arxiv.org/abs/2308.11153
(B.-Kerger-Jiang-Molinaro, Math. Prog., Series B 2024, IPCO 2023)

THANK YOU !

Questions/Comments ?