MASS ASPECT FUNCTION FOR WEAKLY REGULAR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Motivation: (Riemannian) positive mass conjecture

(M^n, g) is **asymptotically Euclidean** if $g \to g_{Eucl}$ at infinity.

(M^n, g) is **asymptotically hyperbolic** if $g \to g_{hyp}$ at infinity.

In both cases the **mass** is an "invariant at infinity that one can use to tell the manifold and the model apart".

Positive mass conjecture:

- A (complete) asymptotically Euclidean manifold such that $Scal(g) \geq 0$ has nonnegative mass. The mass is zero if and only if it is isometric to Euclidean space.
- A (complete) asymptotically hyperbolic n-manifold such that $Scal(g) \geq -n(n-1)$ has nonnegative mass. The mass is zero if and only if it is isometric to hyperbolic space.

Proofs of positive mass conjecture (a sample)

Asymptotically Euclidean:

Schoen-Yau (1979), Witten (1981), Lee-LeFloch (2014),
Bray-Kazaras-Khuri-Stern (2019), Agostiniani-Mazzieri-Oronzio (2021),
Lesourd-Unger-Yau (2021)...

Asymptotically hyperbolic:

X. Wang (2001), Chruściel-Herzlich (2004), Andersson-Cai-Galloway (2007),
Chruściel-Galloway-Nguyen-Paetz (2018), Chruściel-Delay (2019), S. (2021),
Lundberg (2023)...

Asymptotically hyperbolic mass vs asymptotically Euclidean mass

Asymptotically Euclidean mass is a **number** (ADM mass).

In contrast, asymptotically hyperbolic mass is often viewed as a vector or a **functional**.

One explanation for this has to do with the (conformal) infinity: point vs sphere.

Another has to do with the dynamical picture and Noether's theorem. Isometries of $\mathbb{R}^{n,1}$ preserving the model slice: time translations vs time translations and boosts.

Mass aspect function for weakly regular asymptotically hyperbolic manifolds (outline)

1. Overview of mass of asymptotically hyperbolic manifolds:
   - Mass aspect function of Wang (conformal compactification)
   - Mass functional of Chruściel and Herzlich (ADM style)
   - Definition using Ricci tensor
2. Our work (Gicquaud-S. tbp): ADM style definition of mass aspect function (very general asymptotics, low regularity)
   - Mass functional in low regularity
   - Mass aspect function as a distribution on the sphere at infinity
   - Coordinate covariance
3. Outlook: positive mass theorems

Hyperbolic space

We will use the **ball model**:

$$\mathbb{H}^n = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}, \quad \underbrace{b = \rho^{-2} g_{Eucl}}_{= g_{hyp}}, \quad \rho = \frac{1 - |x|^2}{2}.$$

The linear space $\mathscr{N}$ of solutions of

$$Hess\, V - (\Delta V)\, b + (n-1) V\, b = 0$$

(call them **lapses**) is spanned by the functions

$$\frac{1-\rho}{\rho}, \quad \frac{x^1}{\rho}, \quad \dots, \quad \frac{x^n}{\rho}$$

where $x^1, \dots, x^n$ are coordinate functions of $\mathbb{R}^n$.

AH mass functional (Chruściel&Herzlich 2003)

A complete Riemannian manifold (M^n, g), $g \in C^1$, $C^{-1} b \leq g \leq Cb$, is **asymptotically hyperbolic** if

- There is a diffeomorphism $\Phi : M \setminus K \to \mathbb{H}^n \setminus K'$, K and K' are compact.
- $e = \Phi_* g - b$ satisfies

$$\int_{\mathbb{H}^n \setminus K'} \rho^{-1} (|\nabla e|_b^2 + |e|_b^2)\, d\mu^b < \infty \qquad \text{(e.g.} \quad e = O(\rho^\tau), \tau > \tfrac{n}{2}).$$

The **mass functional** is

$$H_\Phi(V) = \lim_{r \to 1} \int_{S_r(0)} [V(div\, e - d(tr\, e) + (tr\, e)dV - e(\nabla V, \cdot)] (\nu) dS$$

where $V \in \mathcal{N} = \mathrm{span}\left\{ \frac{1-\rho}{\rho}, \frac{x^1}{\rho}, \ldots, \frac{x^n}{\rho} \right\}$.

It is well-defined if $\rho^{-1}(Scal + n(n-1)) \in L^1$.

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Mass aspect function (X. Wang 2001)

Assume that (M^n, g) is asymptotically hyperbolic and that in addition

$$e := \Phi_* g - b = \overline{e}\rho^n + h.o.t.$$

where $\overline{e}$ is a smooth 2-tensor on $\overline{B_1}(0)$ which is **transverse**, i.e. $\overline{e}_{ij}x^j \equiv 0$.

In particular, $|e|_b = O(\rho^n)$.

Then

$$H_\Phi\left(\frac{1-\rho}{\rho}\right) = \int_{S_1(0)} \mathrm{tr}^\delta \overline{e}\, d\mu^\sigma, \quad H_\Phi\left(\frac{x^i}{\rho}\right) = \int_{S_1(0)} x^i \mathrm{tr}^\delta \overline{e}\, d\mu^\sigma.$$

Thus, the "mass content" of (M^n, g) is encoded in the **mass aspect function**

$$m := \mathrm{tr}^\delta \overline{e}.$$

Note: m has covarince properties (see e.g. Cortier-Dahl-Gicquaud 2016)

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Mass via Ricci tensor (Herzlich 2015)

Let X be a conformal Killing vector field of $\mathbb{H}^n$ i.e. $\mathscr{L}_X b = \frac{2}{n}(div\, X)b$ which is not Killing.

Then $div\, X = -nV$ for $V \in \mathscr{N}$.

The functional

$$\widetilde{H}(X) := \lim_{r\to 1} \int_{S_r(0)} \left[Ric^g - \frac{1}{2} Scal^g g - \frac{(n-1)(n-2)}{2} g \right] (X, \nu) dS$$

satisfies

$$\widetilde{H}(X) = -nH(V).$$

Our goals/results (Gicquaud-S. tbp):

Give an ADM-style definition of mass aspect function, for general asymptotics and in low regularity, in two steps:

(1) Define mass functional in low regularity:

$$\Phi_* g - b \in W^{1,2}_{1/2} \cap L^\infty_{loc}$$

using cut-off functions.

(2) Use this definition to define mass aspect function for weakly regular asymptotically hyperbolic manifolds by allowing for more general ”test functions”.

Mass functional in low regularity

Let (M,g) be such that $C^{-1}b \leq g \leq Cb$, $e = \Phi_* g - b \in W^{1,2}_{1/2}$, that is we assume

$$\int_{\mathbb{H}^n \setminus K'} \rho^{-1}(|De|_b^2 + |e|_b^2) d\mu^b < \infty$$

as before but g is no longer C^1. In this case *Scal* is defined in the sense of distributions (cf. Lee-LeFloch 2014). For simplicity, we assume that it satisfies $Scal + n(n-1) \in L^1_{-1}$.

Then the mass functional

$$H_\Phi(V) = \lim_{r\to 1} \int_{S_r(0)} [V(div\, e - d(tr\, e)) + (tr\, e)dV - e(\nabla V, \cdot)](\nu) dS, \quad V \in \mathcal{N}$$

has potentially ill-defined terms.

Mass functional in low regularity

The ill-defined boundary terms in

$$H_\Phi(V) = \lim_{r\to 1} \int_{S_r(0)} [V(div\, e - d(tr\, e)) + (tr\, e)dV - e(\nabla V, \cdot)](\nu) dS, \quad V \in \mathcal{N}$$

can be “replaced” using cut-off functions.

Let $(\chi_k)_k$ be a family of compactly supported functions $\chi_k : \mathbb{H}^n \to [0,1]$ such that

- $|\nabla \chi_k| < C$,
- $\Omega_k = \chi_k^{-1}(1)$ are increasing,
- $\cup_k \Omega_k = \mathbb{H}^n$.

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Mass functional in low regularity

For $e = \Phi_* g - b \in W^{1,2}_{1/2} \cap L^\infty$, the "usual" mass functional

$$H_\Phi(V) = \lim_{r \to 1} \int_{S_r(0)} [V(\operatorname{div} e - d(\operatorname{tr} e)) + (\operatorname{tr} e)dV - e(\nabla V, \cdot)](\nu) dS$$

can be replaced by

$$P_\Phi(V) = \lim_{k \to \infty} \int_{\mathbb{H}^n} [V(\operatorname{div} e - d(\operatorname{tr} e)) + (\operatorname{tr} e)dV - e(\nabla V, \cdot)](-\nabla \chi_k) d\mu^b.$$

If $Scal^g + n(n-1) \in L^1_{-1}$ then P_Φ is well-defined and independent of $(\chi_k)_k$.

When $g \in C^1$ we have $P_\Phi(V) = H_\Phi(V)$ for all $V \in \mathcal{N}$.

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Idea of the proof:

Roughly the same argument as in the “classical” case but with cut-off functions.

The first variation of scalar curvature near b reads:

$$Scal^g = -n(n-1) + (n-1)tr(e) + \nabla_i \left[g^{jl} g^{ik} \left(\nabla_j e_{kl} - \nabla_k e_{jl} \right) \right] + Q(e, \nabla e).$$

Here $Q(e, \nabla e)$ is a quadratic term that falls off fast.

Multiply this by $\chi_k V$ and integrate by parts. Avoid introducing $\nabla\nabla\chi_k$.

Use $Hess\, V - (\Delta V)\, b + (n-1)\, V\, b = 0$.

Mass functional: allowing for more general Vs

One can show that

$$P_\Phi(V) = \lim_{k\to\infty} \int_{\mathbb{H}^n} [V(div\, e - d(tr\, e)) + (tr\, e) dV - e(\nabla V, \cdot)] (-\nabla \chi_k)\, d\mu^b$$

is well-defined not only on

$$\mathcal{N} = \{ V : Hess\, V - (\Delta V)\, b + (n-1)\, V\, b = 0 \}$$

but more generally on

$$\widetilde{\mathcal{N}} = \{ V : -\Delta V + nV = 0, \quad Hess\, V - V\, b = O(\rho) \}$$

assuming $e \in L^2_\delta$, $\delta > \frac{n-3}{2}$ (additional restriction if $n > 3$).

This allows us to define mass aspect function as a distribution on $S_1(0)$.

Mass aspect function as a distribution on $S_1(0)$

Theorem

1. *Given any* $v_0 \in C^2(S_1(0))$ *there is a unique* $V \in C^\infty(\mathbb{H}^n)$ *such that*

$$-\Delta V + nV = 0, \quad \rho V_{|S_1(0)} = v_0, \quad Hess\, V - V b = O(\rho).$$

2. *The map* $p_\Phi : C^2(S_1(0)) \to \mathbb{R}$ *defined by* $p_\Phi(v_0) = P_\Phi(V)$ *is continuous. When g is asymptotically hyperbolic in the sense of X. Wang with mass aspect function m we have*

$$p_\Phi(v_0) = n \int_{S_1(0)} m\, v_0\, d\mu^\sigma.$$

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The definition of mass aspect using Ricci tensor

In a similar spirit, one can define (assuming a bit more regularity):

$$\widetilde{p}(v_0) := \lim_{k\to\infty} \int_{\mathbb{H}^n} \left[Ric^g - \frac{1}{2} Scal^g g - \frac{(n-1)(n-2)}{2} g \right] (\nabla V, -\nabla \chi_k)\, d\mu^b$$

where V is such that

$$-\Delta V + nV = 0, \quad \rho V_{|S_1(0)} = v_0, \quad Hess\, V - V\, b = O(\rho).$$

Here $X = \nabla V$ replaces a conformal Killing vector field as it will satisfy $\mathscr{L}_X b - \frac{2}{n}(div\, X) b = O(\rho)$.

Dependence of the mass functional on the chart

Suppose we have two charts at infinity $\Phi_i : M \setminus K \to \mathbb{H}^n \setminus K'_i$, $i = 1, 2$.

We want to ensure (in low regularity):

1. **Asymptotic rigidity:** under suitable assumptions on (M, g) and Φ_i, the transition map $\Phi_2 \circ \Phi_1^{-1} : \mathbb{H}^n \setminus K'_1 \to \mathbb{H}^n \setminus K'_2$ satisfies

$$\Phi_2 \circ \Phi_1^{-1} = A \circ \Phi_0 \tag{1}$$

   for $A \in O_0(n, 1)$ and $\Phi_0(x) = exp_x(\xi(x))$ "asymptotic to identity".

2. **Covariance:** if (1) holds then the mass functional satisfies

$$P_{\Phi_2}(V) = P_{\Phi_1}(V \circ A^{-1}).$$

Regularity assumptions and comments

Need to assume that $\Phi_i^* g - b \in W^{k,p}_\delta$ with

1. $kp > n$ to ensure $W^{k,p}_\delta \hookrightarrow W^{1,q}_\delta$ with $q > n$
2. $\delta + \frac{n-1}{p} \geq 2$: this condition has an intrinsic interpretation

This argument is similar to the one used by Bartnik 1986 to prove asymptotic rigidity of asymptotically Euclidean manifolds.

A "smooth" version of this argument for asymptotically hyperbolic manifolds was outlined by Herzlich 2005.

An issue with low regularity is that we need to work with integrals and bump functions to achieve our goals.

Covariance

A version of “miraculous cancellation” (Bartnik 1986).

Write the mass functional as

$$P(e,V) = \lim_{k\to\infty} \int_{\mathbb{H}^n} [V(div\, e - d(tr\, e)) + (tr\, e)dV - e(\nabla V, \cdot)](-\nabla \chi_k) d\mu^b.$$

If we have charts Φ_1 and Φ_2 at infinity such that $\Phi_2 \circ \Phi_1^{-1}(x) = \exp_x(\xi(x))$ then

$$P(e_1, V) - P(e_2, V) = P(\mathscr{L}_\xi b, V) = \lim_{k\to\infty} \int_{\mathbb{H}^n} \mathrm{div}\, \mathbb{U}_k d\mu^b$$

where $e_i = (\Phi_i)_* g - b$ as before and $\mathbb{U}_k$ are compactly supported.

Thus $P(e_1, \cdot) = P(e_2, \cdot)$ i.e. mass functionals are the same.

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Outlook: positive mass theorems

With the above definition of mass one could possibly relax the regularity assumptions in the available asymptotically hyperbolic positive mass theorems.

The positivity part would follow by a smoothing argument and continuity of the mass functional.

But the rigidity part would require a separate argument (as usual).

The ultimate goal would be to get a better understanding of positivity of mass on the level of mass aspect function.

THANK YOU!