Non-existence of extremals for the second conformal eigenvalue of the conformal laplacian in small dimensions

CRM Workshop "Analysis of Geometric Singularities"

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Current Section

The Conformal Laplacian

Throughout this talk (M^n, g) will be a closed manifold of dimension $n \geq 3$.
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$$L_g = \triangle_g + \frac{n-2}{4(n-1)} S_g$$

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This operator is conformally invariant: if $u \in C^\infty(M), u > 0$, and $g_u = u^{\frac{4}{n-2}} g$,

$$L_g(uf) = u^{\frac{n+2}{n-2}} L_{g_u}(f) \quad \text{for any } f \in C^\infty(M).$$

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In particular:

$$\int_M f L_{g_u} f dv_{g_u} = \int_M (uf) L_g(uf) dv_g.$$

Conformal eigenvalues

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For $k \geq 1$ we define the k-th conformal eigenvalue of L_g as:

$$\Lambda_k(M, [g]) = \inf_{\tilde{g} \in [g]} \left(\lambda_k(\tilde{g}) \mathrm{Vol}(M, \tilde{g})^{\frac{2}{n}} \right).$$

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$$\Lambda_k(M, [g]) = \inf_{\substack{u \in C^\infty(M) \\ u > 0}} \lambda_k(g_u) \Big(\int_M u^{\frac{2n}{n-2}} dv_g \Big)^{\frac{2}{n}}.$$

Generalities

By [Ammann-Jammes '08], for any $k \geq 1$,

$$\sup_{\substack{u \in C^\infty(M) \\ u > 0}} \lambda_k(g_u) \mathrm{Vol}(M, g_u)^{\frac{2}{n}} = +\infty.$$

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A remark: our $\Lambda_k(M, [g])$ are an infimum, and this creates big conceptual difference with the maximisation problem of conformal eigenvalues of the Laplacian in dimensions $n \geq 2$ ([Nadirashvili-Sire '15], [Pétrides '18, '22], [Karpukhin-Stern '20, '22]). We will restrict to the conformal case here.

Current Section

The Yamabe problem

Recall the definition of the Yamabe invariant of $[g]$:

$$Y(M,[g]) = \inf_{f \in C^\infty(M)\backslash\{0\}} \frac{\int_M f \cdot L_g f dv_g}{\left(\int_M |f|^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}} > 0.$$

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Investigated by [Yamabe '60], [Trudinger '68], [Aubin '76], [Schoen '84]. It turns out that the Yamabe invariant is equal to the first conformal eigenvalue $\Lambda_1(M,[g])$:

$$Y(M,[g]) = \Lambda_1(M,[g]).$$

The Yamabe problem – proof that $Y(M,[g]) = \Lambda_1(M,[g])$.

Proof.

Since $dv_{g_u} = u^{\frac{2n}{n-2}} dv_g$ we have, letting $h = uf$,

$$\lambda_1(g_u) = \inf_{f \in C^\infty(M)\setminus\{0\}} \frac{\int_M f L_{g_u} f dv_{g_u}}{\int_M f^2 dv_{g_u}}$$

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Choose $h = u$ in $\lambda_1(g_u)$. Then

$$\Lambda_1(M,[g]) \le \lambda_1(g_u) \Big(\int_M u^{\frac{2n}{n-2}} dv_g \Big)^{\frac{2}{n}}$$

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The other inequality follows from Hölder's inequality:

$$\int_M u^{\frac{4}{n-2}} f^2 dv_g \le \Big(\int_M u^{\frac{2n}{n-2}} dv_g\Big)^{\frac{2}{n}} \Big(\int_M |f|^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}}.$$

The Yamabe problem - extremals

[Aubin '76] showed that if

$$\Lambda_1(M,[g]) < \Lambda_1(\mathbb{S}^n,[g_0]) \qquad (*)$$

where g_0 is the round metric, then $\Lambda_1(M,[g])$ is attained. He showed $(*)$ when $n \geq 6$ and (M,g) is not l.c.f.

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Minimisers attaining $\Lambda_1(M, [g])$ (or $Y(M, [g])$) are positive least-energy solutions of the Yamabe equation:

$$L_g u = u^{\frac{n+2}{n-2}} \quad \text{in } M$$

with

$$L_g = \triangle_g u + \frac{n-2}{4(n-1)} S_g.$$

Current Section

Attaining $\Lambda_2(M,[g])$.

[Ammann-Humbert '06] showed that if

$$\Lambda_2(M,[g])^{\frac{n}{2}} < \Lambda_1(M,[g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n,[g_0])^{\frac{n}{2}} \qquad (**)$$

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We address here the low-dimensional case $n \leq 10$. Our main result states can $(**)$ can no longer be expected to hold in general.

Nonexistence of extremals for $\Lambda_2(M, [g])$ when $n \leq 10$

Our main result shows that, in dimensions $n \leq 10$, metrics in $\mathbb{S}^n$ that are close enough to the round metric do not attain Λ_2:

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Theorem (P.-Vétois, '24)

Assume that $3 \leq n \leq 10$. There exists $m \in \mathbb{N}^*$ and $\delta > 0$ such that for every smooth metric g in $\mathbb{S}^n$ with $\|g - g_0\|_{C^m} < \delta$ we have

$$\Lambda_2(\mathbb{S}^n, [g])^{\frac{n}{2}} = \Lambda_1(\mathbb{S}^n, [g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n, [g_0])^{\frac{n}{2}}$$

and $\Lambda_2(\mathbb{S}^n, [g])^{\frac{n}{2}}$ is not attained.

Remarks on our main result

- [Ammann-Humbert '06] showed that $\Lambda_2(\mathbb{S}^n, [g_0])^{\frac{n}{2}} = 2\Lambda_1(\mathbb{S}^n, [g_0])^{\frac{n}{2}}$ and that it is never attained.

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- Our Theorem is also the first non-existence result of extremals for conformal eigenvalues on manifolds that are not standard spheres (in all contexts).
- This result shows a striking difference with the $n \geq 11$ case: when $n \leq 10$, one cannot guarantee anymore that $\Lambda_2(M, [g])$ is attained solely by enforcing local conditions on g.

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- Our Theorem is also the first non-existence result of extremals for conformal eigenvalues on manifolds that are not standard spheres (in all contexts).
- This result shows a striking difference with the $n \geq 11$ case: when $n \leq 10$, one cannot guarantee anymore that $\Lambda_2(M, [g])$ is attained solely by enforcing local conditions on g. New ideas are needed to produce examples of manifolds of dimension $n \leq 10$ where $\Lambda_2(M, [g])$ is attained, with or without equality for $\Lambda_2(M, [g])$.

Current Section

Variational characterisation of $\Lambda_k(M,[g])$

For $u \in C^\infty(M), u > 0$ let $g_u = u^{\frac{4}{n-2}} g$. The classical variational characterisation of $\lambda_k(g)$ and conformal invariance show that:

$$\lambda_k(u) = \lambda_k(g_u) = \inf_{\substack{\dim V = k \\ V \subset H^1(M)}} \sup_{v \in V \setminus \{0\}} \frac{\int_M v L_{g_u} v dv_{g_u}}{\int_M v^2 dv_{g_u}}$$

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$$L_g \varphi = \lambda_k(u) u^{\frac{4}{n-2}} \varphi \quad \text{in } M, \quad \int_M u^{\frac{4}{n-2}} \varphi^2 dv_g = 1.$$

Extremals for $\Lambda_k(M,[g])$

Definition

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When $k=1$, we saw that $\Lambda_1(M,[g]) = Y(M,[g])$.

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$\Lambda_2(M, [g])$ may only be attained at a generalised metric $g_u = u^{\frac{4}{n-2}} g$, which are singular at $\{u = 0\}$.

Euler-Lagrange equation for $\Lambda_2(M, [g])$

It is proven in [Ammann-Humbert '06], [Gursky-Perez Ayala '21] that IF $\Lambda_2(M, [g])$ is attained at $u \in L^{\frac{2n}{n-2}}(M) \backslash \{0\}$, $u \geq 0$ a.e. in M then $\lambda_2(u)$ is simple, that is spanned by a single non-zero normalised eigenfunction φ which satisfies

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$\Lambda_2(M,[g])$ is thus a natural generalisation of the Yamabe problem: attaining it provides sign-changing solutions of least energy, whereas attaining $\Lambda_1(M,[g])$ provided solutions of least-energy (thus positive).

Current Section

Reduction to a PDE proof

Theorem (The result we want to prove)

Assume that $3 \leq n \leq 10$. *There exists* $m \in \mathbb{N}$ *and* $\delta > 0$ *such that for every smooth metric* g *in* $\mathbb{S}^n$ *with* $\|g - g_0\|_{C^m} < \delta$ *we have*

$$\Lambda_2(\mathbb{S}^n, [g])^{\frac{n}{2}} = \Lambda_1(\mathbb{S}^n, [g])^{\frac{n}{2}} + \Lambda_1(\mathbb{S}^n, [g_0])^{\frac{n}{2}}$$

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By contradiction: assume that there exists a sequence $(g_k)_{k \geq 0}$ of smooth metrics in $\mathbb{S}^n$, converging to g_0 in C^m, for which every $\Lambda_2(\mathbb{S}^n, [g_k])$ is attained.

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$$L_{g_k}\varphi_k = |\varphi_k|^{\frac{4}{n-2}}\varphi_k \quad \text{in } M \quad \text{and} \quad \int_M |\varphi_k|^{\frac{2n}{n-2}} dv_{g_k} = \Lambda_2(\mathbb{S}^n, [g_k])^{\frac{n}{2}}.$$

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We find a contradiction using the minimality of φ_k.

Weak convergence towards a union of two spheres

By definition of φ_k and since $g_k \to g_0$ in C^m,

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where $\int_{\mathbb{S}^n} B_{i,k}^{\frac{2n}{n-2}} dv_{g_0} = \Lambda_1(\mathbb{S}^n, [g_0])^{\frac{n}{2}}$,

$$B_{i,k}(x) \approx \frac{\mu_{i,k}^{\frac{n-2}{2}}}{\left(\mu_{i,k}^2 + d_g(x_{i,k}, x)^2\right)^{\frac{n-2}{2}}}, \quad x \in M, \quad \mu_{i,k} \le 1$$

and

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We then prove quantitative estimates on each sphere: something like

$$\left|\varphi_k - \left(B_{1,k} - B_{2,k}\right)\right| \lesssim \left(\sum_{\alpha=2}^{\left[\frac{n-2}{2}\right]} \varepsilon_{\alpha,k} d_{g_0}(x_{i,k}, \cdot)^{2\alpha}\right) B_{i,k},$$

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The assumption $n \leq 10$ is crucial here: it forces each profile $B_{i,k}$ to concentrate as $k \to +\infty$ as follows:

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for $i = 1, 2$. Follows from the techniques in [P. '22], [P.-Vétois '22].

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$$\frac{\mu_{1,k}\mu_{2,k}}{\mu_{1,k}^2 + d_{g_0}(x_{1,k}, x_{2,k})^2} = o\left(\mu_{2,k}^2\right),$$

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End of the proof when $6 \leq n \leq 10$

When $6 \leq n \leq 10$ the convergence $\Lambda_2(\mathbb{S}^n, [g_k])^{\frac{n}{2}} \to 2\Lambda_1(\mathbb{S}^n, [g_0])^{\frac{n}{2}}$ and the smooth pointwise estimates on φ_k allow us to estimate:

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$$\Lambda_1(\mathbb{S}^n, [g_k]) - \Lambda_1(\mathbb{S}^n, [g_0]) \geq -C \sum_{\alpha=2}^{[\frac{n-2}{2}]} \varepsilon_{\alpha,k} \mu_{1,k}^{2\alpha} \qquad (\dagger)$$

for some $C > 0$ independent of k, where $\varepsilon_{\alpha,k} \sim \|\nabla^\alpha (g_k - g_0)\|_\infty^2$.

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which contradicts $(\dagger)$ for $\theta >> 1$.

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This argument crucially uses that $\mu_{1,k} \to 0$, which is only guaranteed when $3 \leq n \leq 10$ [P.-Vétois, '22].

Thank you for your attention.

Euler-Lagrange equation for $\Lambda_2(M,[g])$

Let

$$F(u) = \lambda_2(g_u)\Big(\int_M u^{\frac{2n}{n-2}}dv_g\Big)^{\frac{2}{n}}.$$

If $u_t = u(1+th)$ for $|t|$ small enough we have

$$\frac{d}{dt}_{|t=0_+}F(u) = \inf_{\varphi\in E_2(u)}\left(-(2^*-2)\lambda_2(g_u)\frac{\int_M u^{2^*-2}h\varphi^2dv_g}{\int_M u^{2^*-2}\varphi^2dv_g}\right) + (2^*-2)\lambda_2(g_u)\int_M u^{2^*}hdv_g$$

$$\frac{d}{dt}_{|t=0_-}F(u) = \sup_{\varphi\in E_2(u)}\left(-(2^*-2)\lambda_2(g_u)\frac{\int_M u^{2^*-2}h\varphi^2dv_g}{\int_M u^{2^*-2}\varphi^2dv_g}\right) + (2^*-2)\lambda_2(g_u)\int_M u^{2^*}hdv_g$$

The fact that u is a minimiser and $\lambda_2(g_u)>0$ implies $\dim E_2(u)=1$.

Euler-Lagrange equation for $\Lambda_k(M,[g])$, $k \geq 2$

Let

$$F(u) = \lambda_k(g_u)\Big(\int_M u^{\frac{2n}{n-2}} dv_g\Big)^{\frac{2}{n}}$$

and assume u attains $\inf_{u>0} F(u)$.

Then there exists $\varphi_1, \dots, \varphi_k$ generalised eigenvectors associated to $\lambda_k(u)$ and $d_1, \dots, d_k$ nonnegative numbers with $\sum_{i=1}^k d_i = 1$ such that

$$u^2 = \sum_{i=1}^k d_i \varphi_i^2 \quad \text{in } M.$$

Conformal eigenvalues in different contexts

That minimisers of $\Lambda_2(M,[g])$ have one-dimensional eigenspaces is deeply related to the minimisation problem and to the fact that there is a spectral gap.

The situation is very different from the maximisation of the eigenvalues of the laplacian in [Nadirashvili-Sire '15], [Pétrides '18, '22], [Karpukhin-Stern '20, '22]. There extremals in a conformal class yield in general harmonic maps into some sphere by eigenfunctions and extremals over all metrics (in dimension 2) yield minimal immersions into some sphere. This is likely to be the case for extremals of $\Lambda_k(M,[g])$, $k \geq 3$.

On manifolds with negative Yamabe invariant (so $\Lambda_1(M,[g]) < 0$): the second negative eigenvalue of L_g can be maximised in a conformal class [Gursky-Pérez Ayala '21]. Extremals u yield either a sign-changing solutions of Yamabe or harmonic mappings from $M \backslash \{u = 0\}$ into a sphere.