Elliptic systems with critical growth

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The problem

$$\text{(S)} \qquad -\Delta u_i + \lambda_i u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \quad \text{in } \mathbb{R}^n, \quad u_i > 0 \text{ in } \mathbb{R}^n, \quad i = 1, \dots, m,$$

- $1 < p \leq 2^* - 1$, $2^* := \frac{2n}{n-2}$, $n \geq 3$
- $\beta_{ij} = \beta_{ji} \in \mathbb{R}$
- $\lambda_i \in \mathbb{R}$
- $m \in \mathbb{N}$

The Gross-Pitaevskii system

$$(S) \qquad -\Delta u_i + \lambda_i u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \quad \text{in } \mathbb{R}^n, \quad u_i > 0 \text{ in } \mathbb{R}^n, \quad i = 1, \dots, m,$$

The complex valued function $\Phi_i(t,x) = e^{\iota \lambda_i t} u_i(x)$ is a solitary wave solution of

$$-\iota \partial_t \Phi_i = \Delta \Phi_i + \sum_{j=1}^{m} \beta_{ij} |\Phi_j|^{\frac{p+1}{2}} |\Phi_i|^{\frac{p-1}{2}}, \ i = 1, \dots, m$$

- $|u_i|$ is the amplitude of the $i-$th density
- $\beta_{ii} = \mu_i$ describe the interaction between particles of the same component
- β_{ij}, $i \neq j$ describe the interaction between particles of different components

$\beta_{ij} > 0 \ \rightsquigarrow$ attractive or cooperative interaction

$\beta_{ij} < 0 \ \rightsquigarrow$ repulsive or competive interaction

The critical case

$$(S) \qquad -\Delta u_i + \lambda_i u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \quad \text{in } \mathbb{R}^n, \quad i = 1, \dots, m,$$

- We consider the case $\beta_{ii} := \mu_i > 0$
- The subcritical case, i.e. $p < \frac{n+2}{n-2}$ has been widely studied: Ambrosetti, Bartsch, Byeon, Clapp, Colorado, Dancer, Du, T-C Lin, Z. Liu, Wei, Maia, Montefusco, Pellacci, Sato, Sirakov, Soave, Szulkin, Tavares, Terracini, Verzini, Z-Q Wang, Weth, Z. Zhang, etc.
- We will focus on the critical case, i.e. $\boxed{p = \frac{n+2}{n-2}}$.

Preliminaries

Different kind of solutions

$$\text{(S)} \qquad -\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \quad \text{in } \mathbb{R}^n, \quad u_i > 0 \text{ in } \mathbb{R}^n, \qquad i = 1, \ldots, m,$$

- A trivial solution has all trivial components, i.e. $u_i \equiv 0$ for any i
- A non-trivial solution has some trivial components, i.e. $u_i \equiv 0$ for some i
  - For example, if $u_2 = \cdots = u_m = 0$ the system reduces to the critical equation

$$-\Delta u_1 = \mu_{11} u_1^p \quad \text{in } \mathbb{R}^n$$

- A fully non-trivial solution has all non-trivial components, i.e. $u_i \not\equiv 0$ for any i

Two kind of fully non-trivial solutions

If we look for fully non-trivial solutions to the system

$$(S) \qquad -\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \quad \text{in } \mathbb{R}^n, \quad u_i > 0 \text{ in } \mathbb{R}^n, \quad i = 1, \ldots, m,$$

we can find two kind of solutions:

- synchronized solutions
- non-synchronized solutions

Synchronized solutions

The single equation

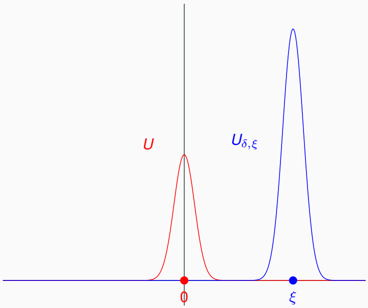
It is well known that the problem

$$\boxed{-\Delta U = U^p \text{ in } \mathbb{R}^n}$$

has a infinitely many positive solutions, i.e. the so-called *bubbles*

$$U_{\delta,\xi}(x) = \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{x-\xi}{\delta}\right), \quad U(x) = \alpha_n \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}.$$

The bubbles U



Synchronized solutions

We say that $\mathbf{u} = (u_1, \ldots, u_m)$ is a synchronized solution to the system:

$$(S) \quad -\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \quad \text{in } \mathbb{R}^n, \quad i = 1, \ldots, m$$

if $\mathbf{u} = (s_1 U, \ldots, s_m U)$ with $s_i > 0$ where U is a positive solution to the critical equation $-\Delta U = U^p$ in $\mathbb{R}^n$

The system (S) has a synchronized solution

$\Updownarrow$

The algebraic system

$$s_i = \sum_{j=1}^{m} \beta_{ij} s_j^{\frac{p+1}{2}} s_i^{\frac{p-1}{2}}, \ i = 1, \ldots, m$$

has a solution $s_1 > 0, \ldots, s_m > 0$

Existence of synchronized solutions

$$(S)\qquad -\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \ \text{in } \mathbb{R}^n, \quad u_i > 0 \text{ in } \mathbb{R}^n, \quad i = 1, \dots, m,$$

has a syncronized solution if

- $p = 3$ (i.e. $n = 4$) $^{(*)}$: $\beta_{ij} = \beta$, for all $i \neq j$ and $\beta \in (-\beta^*, \min\{\beta_{ii}) \cup (\max \beta_{ii}, +\infty)$
- $p < 3$ (i.e. $n \geq 5$) $^{(**)}$: $\beta_{ij} > 0$ for all i, j

Proof.

- Solutions of the algebraic system

$$s_i = \sum_{j=1}^{m} \beta_{ij} s_j^{\frac{p+1}{2}} s_i^{\frac{p-1}{2}}, \ i = 1, \dots, m$$

are critical points of the function

$$J(\mathbf{s}) := \frac{1}{2} \sum_{i=1}^{m} s_i^2 - \frac{1}{p+1} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \beta_{ij} s_j^{\frac{p+1}{2}} s_i^{\frac{p+1}{2}}, \ \mathbf{s} := (s_1, \dots, s_m) \in \mathbb{R}^m$$

on the Nehari manifold

$$\mathcal{N} := \left\{ \mathbf{s} \in \mathbb{R}^m \setminus \{0\} \ : \ \langle \nabla J(\mathbf{s}), \mathbf{s} \rangle = 0 \right\}$$

- There exists a minimum point since

$$J(\mathbf{s}) = \frac{2}{n} \sum_{i=1}^{m} s_i^2, \ \mathbf{s} \in \mathcal{N}$$

- if $p < 3$ and $\beta_{ij} > 0$ all the components of the minimum point are strictly positive, i.e. $s_i > 0$.

$^{(*)}$ Bartsch (2013), $^{(**)}$ Clapp & Pistoia (2020)

Cooperative systems can only have syncronized solutions!

In the cooperative case, i.e. $\beta > 0$, the only positive solution to the system

$$\begin{cases} -\Delta u = u^p + \beta u^{\frac{p-1}{2}} v^{\frac{p+1}{2}} \text{ in } \mathbb{R}^n, \\ -\Delta v = v^p + \beta u^{\frac{p+1}{2}} v^{\frac{p-1}{2}} \text{ in } \mathbb{R}^n, \\ u, v \in D^{1,2}(\mathbb{R}^n), \end{cases}$$

is radially symmetric and synchronized.

- Guo-Liu (2008)

The only positive solution to the system

$$-\Delta u_i = (u_1^2 + \cdots + u_m^2)^{\frac{p-1}{2}} u_i \text{ in } \mathbb{R}^n, \ u_i \in D^{1,2}(\mathbb{R}^n), \ i = 1, \dots, m,$$

is radially symmetric and synchronized.

- Druet & Hebey (2009), Druet, Hebey & Vetois (2010)

- If $n = 4$, $p = 3$ and system reduces to the cooperative system

$$-\Delta u_i = (u_1^2 + \cdots + u_m^2) u_i \text{ in } \mathbb{R}^4, \ i = 1, \dots, m$$

Segregated solutions

Positive solutions to the system and sign-changing solutions to the equation

There is a strong relation between sign-changing solutions to the single equation

$$-\Delta w = |w|^{\frac{4}{n-2}} w \quad \text{in } \mathbb{R}^n$$

and positive solutions to the system

$$(S_\beta) \qquad \begin{cases} -\Delta u = u^p + \beta u^{\frac{p-1}{2}} v^{\frac{p+1}{2}} \text{ in } \mathbb{R}^n, \\ -\Delta v = v^p + \beta u^{\frac{p+1}{2}} v^{\frac{p-1}{2}} \text{ in } \mathbb{R}^n, \\ u, v \in D^{1,2}(\mathbb{R}^n), \end{cases}$$

in the strongly competitive case, i.e. $\beta \to -\infty$:

$$u \sim w^+ \text{ and } v \sim w^-$$

Example: the phase separation phenomenon when $\beta \to -\infty$

Let us consider the subcritical system on a bounded domain $\Omega \subset \mathbb{R}^n$ (*)

$$(S_\beta) \quad \begin{cases} -\Delta u = u^p + \beta u^{\frac{p-1}{2}} u^{\frac{p+1}{2}} \text{ in } \Omega, \\ -\Delta v = v^p + \beta v^{\frac{p-1}{2}} u^{\frac{p+1}{2}} \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega \end{cases}$$

- there exists a least energy positive solution (u_β, v_β) which is the minimum of the energy

$$E_\beta(u,v) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) - \frac{1}{p+1}\int_\Omega \left(|u|^{p+1} + |v|^{p+1} + 2\beta |u|^{\frac{p+1}{2}} |v|^{\frac{p+1}{2}}\right)$$

onto the the Nehari manifold

$$\mathcal{N} := \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \;:\; u, v \neq 0,\; \partial_u E_\beta(u,v)u = 0,\; \partial_v E_\beta(u,v)v = 0\},$$

- $u_\beta \sim u_\infty$ & $v_\beta \sim v_\infty$ as $\beta \to -\infty$ and

$$E_\beta(u_\beta, v_\beta) \le c \;\Rightarrow\; \underbrace{-\beta}_{\downarrow \, +\infty} \int_{\mathbb{R}^n} \underbrace{|u_\beta|^{\frac{p+1}{2}} |v_\beta|^{\frac{p+1}{2}}}_{\downarrow \, 0} \le c \quad \Rightarrow \quad \boxed{u_\infty \cdot v_\infty \equiv 0 \text{ in } \mathbb{R}^n}$$

- $w := u_\infty - v_\infty$ is a least energy sign-changing solution to the single equation

$$-\Delta w = |w|^{p-1} w \text{ in } \Omega,\; w = 0 \text{ on } \partial\Omega$$

- $\Omega_1 := \{x \in \Omega \;:\; u_\infty(y) > 0\}$ and $\Omega_2 := \{x \in \Omega \;:\; v_\infty(y) > 0\}$ are a partition of Ω.

(*) Conti-Terracini-Verzini (2002,2003), Caffarelli-Lin (2008), Noris-Tavares-Terracini-Verzini (2010), Soave-Zilio (2015), Chen-Zou (2012,2015)

An idea

Whenever there exists a sign-changing solutions to the single equation

$$-\Delta w = |w|^{p-1} w \text{ in } \mathbb{R}^n$$

hopefully (using the same strategy) one could find a positive solution to the system

$$\begin{cases} -\Delta u = u^p + \beta u^{\frac{p-1}{2}} v^{\frac{p+1}{2}} \text{ in } \mathbb{R}^n, \\ -\Delta v = v^p + \beta u^{\frac{p+1}{2}} v^{\frac{p-1}{2}} \text{ in } \mathbb{R}^n \end{cases}$$

in the competitive case, i.e. $\beta < 0$,

Sign-changing solutions to the single equation

The single equation

(1) $\boxed{-\Delta u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{R}^n}$ $\Longleftrightarrow$ (2) $\boxed{-\Delta_{\mathfrak{g}_0} u + \frac{n(n-2)}{4} u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{S}^n}$

- The stereographic projection $\pi : \mathbb{R}^n \to \mathbb{S}^n \setminus \{S\}$

$$\pi(y) = \left(\frac{2y}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2} \right)$$

is a local conformal diffeomorphism from the euclidean space to the round sphere $(\mathbb{S}^n, \mathfrak{g}_o)$, i.e.

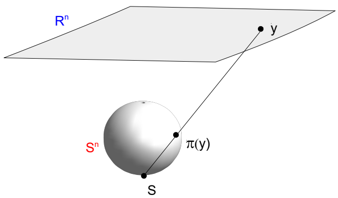
$$\pi^* \mathfrak{g}_o = \phi^{\frac{4}{n-2}} dy, \ \phi(y) := \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}},$$

- Then the following formula holds

$$(\mathcal{L}_{\mathfrak{g}_o} u) \circ \pi = \phi^{-\frac{n+2}{n-2}} \Delta \left(\phi(u \circ \pi) \right), \ u \in H^1(\mathbb{S}^n),$$

- Then

$$U = \phi(u \circ \pi) \text{ solves (1)} \quad \Leftrightarrow \quad u \text{ solves (2)}$$

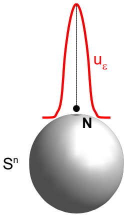


Existence of positive solutions: Obata (1972), Aubin (1976), Talenti (1976)

- On the round sphere $\mathbb{S}^n$ all the positive solutions, up to rotations, are

$$(u_\epsilon \circ \pi)(y) = c_n \epsilon^{\frac{n-2}{2}} \left(\frac{1 + |y|^2}{\epsilon^2 + |y|^2} \right)^{\frac{n-2}{2}} , \quad \epsilon > 0$$

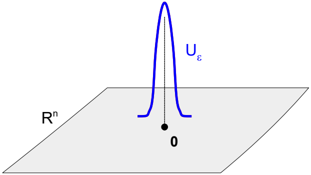
  - u_ϵ blows-up at the north pole as $\epsilon \to 0$
  - $u_1(x) \equiv c_n := \left(\frac{n(n-2)}{4} \right)^{\frac{n-2}{4}}$ is a constant solution



- On the Euclidean space $\mathbb{R}^n$ all the positive solutions, up to traslations, are

$$U_\epsilon(y) = \phi(y)\,(u_\epsilon \circ \pi)(y) = \alpha_n \frac{\epsilon^{\frac{n-2}{2}}}{(\epsilon^2 + |y|^2)^{\frac{n-2}{2}}}, \quad \epsilon > 0$$

  - U_ϵ blows-up at the origin as $\epsilon \to 0$
  - $U_1(y) := \alpha_n \frac{1}{(1+|y|^2)^{\frac{n-2}{2}}}$, $\alpha_n := (n(n-2))^{\frac{n-2}{4}}$



Existence of sign changing solutions Ding (1986)

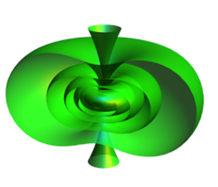
The problem

$$-\Delta_{\mathfrak{g}_0} u + \frac{n(n-2)}{4} u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{S}^n$$

has infinitely many sign-changing solutions, which are invariant under the action of $\mathcal{O}(k) \times \mathcal{O}(n+1-k)$, $k = 2, \dots, n-1$.

Proof.

- The solutions are critical points of the energy $E : H^1(\mathbb{S}^n) \to \mathbb{R}^n$
  $E(u) = \frac{1}{2} \int_{\mathbb{S}^n} \left(|\nabla u|^2 + \frac{n(n-2)}{4} u^2 \right) d\sigma - \frac{1}{p+1} \int_{\mathbb{S}^n} |u|^{p+1} d\sigma$
- $H^1(\mathbb{S}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{S}^n)$ is not compact $\Rightarrow E$ does not satisfy the Palais-Smale condition.
- How to recover the compactness? We use the fact that $\mathbb{S}^N$ enjoys a lot of simmetries!
- Let $\Gamma = \mathcal{O}(k) \times \mathcal{O}(n+1-k) \subset \mathcal{O}(n+1)$
- Let $H^1_\Gamma(\mathbb{S}^n) := \left\{ u \in H^1(\mathbb{S}^n) \ : \ u \text{ is } \Gamma-\text{invariant}, \text{ i.e. } u(\gamma x) = u(x), \ \forall \gamma \in \Gamma, \ x \in \mathbb{S}^n \right\}$
- The critical points of E restricted to $H^1_\Gamma(\mathbb{S}^n)$ are Γ-invariant solutions to the equation
- The Γ-orbit $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ is homeomorphic to $\mathbb{S}^{k-1} \times \mathbb{S}^{n-k}$ or $\mathbb{S}^{k-1}$ or $\mathbb{S}^{n-k}$.
- $1 \leq \dim(\Gamma x) \leq n-1 \Rightarrow H^1_\Gamma(\mathbb{S}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{S}^n)$ is compact.
- The restriction of E to $H^1_\Gamma(\mathbb{S}^n)$ has a sequence of critical points with increasing energy (via Ljusternik-Schnirelman category), which are solutions of the unrestricted problem according to the principle of symmetric criticality (Palais 1979)



The level sets are tori

Existence of sign changing solutions: Del Pino, Musso, Pacard & Pistoia (2011,2013)

The problem

$$-\Delta_{\mathfrak{g}_0} u + \frac{n(n-2)}{4} u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{S}^n$$

has infinitely many sign-changing solutions, which are the superposition of the constant solution with a large number of negative bubbles which blow-up at points which in turn are regularly arranged along some minimal submanifolds of $\mathbb{S}^n$.

- These are not invariant under the action of $\mathcal{O}(2) \times \mathcal{O}(n-1) \Rightarrow$ They are different from Ding's solutions!
- The proof relies on a Ljapunov-Schmidt procedure: the parameter is the large number of negative bubbles!

Concentration on a great circle ($n \geq 3$) on $\mathbb{S}^n$ Del Pino, Musso, Pacard & Pistoia (2013))

- $\mathbb{S}^n \subset \mathbb{C} \times \mathbb{R}^{n-1}$
- $\mathbb{T}^1 := \mathbb{S}^1 \times \{0\}$ is a great circle of $\mathbb{S}^n$

There exists $k_0 > 0$ such that for any $k \geq k_0$ there exists u_k solution to

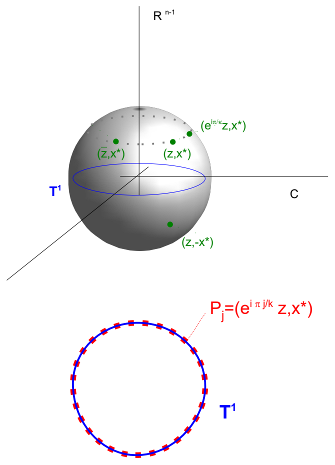
$$-\Delta_{g_0} u + \frac{n(n-2)}{4} u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{S}^n$$

such that u_k has the following invariances

- $u_k(z, x^*) = u_k(\bar{z}, x^*) = u_k(z, -x^*) = u_k\left(e^{\frac{i\pi}{k}} z, x^*\right)$

and as $k \to \infty$

- $u_k \to c_n$ uniformly on compact sets of $\mathbb{S}^n \setminus \mathbb{T}^1$
- u_k blow-up negatively at the $2k$ points $P_j := \left(e^{\frac{\pi j}{k} i}, 0\right) \in \mathbb{T}^1$, $j = 1, \ldots, 2k$



How this solution looks like in $\mathbb{R}^n$ Del Pino, Musso, Pacard & Pistoia (2011)

The problem

$$-\Delta u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{R}^n$$

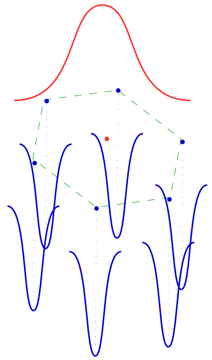
admits infinitely many sign-changing non-radial solutions which are invariant under the action of $\mathcal{D}_k \times \mathcal{O}(n-1)$, where $\mathcal{D}_k$ is the dihedral group of $\mathbb{R}^2$ for k large enough.

More precisely, there exists $k_0 > 0$ such that for any $k \geq k_0$ there exists a solution u_k such that

$$u_k(y) = U(y) - \sum_{\ell=1}^{k} \delta^{-\frac{n-2}{2}} U\left(\frac{y - \xi_\ell}{\delta}\right) + \phi(y)$$

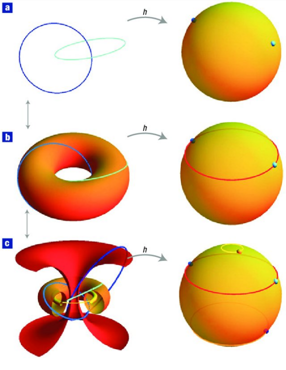
where

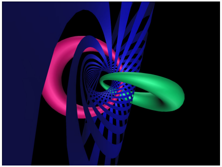
- $U(y) = \alpha_n \frac{1}{(1+|y|^2)^{\frac{n-2}{2}}}$ solves $-\Delta U = U^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$
- the concentration parameter $\delta \sim \frac{1}{k^2}$ if $n \geq 4$, $\frac{1}{(k \log k)^2}$ if $n = 3$
- the concentration points $\xi_\ell \sim \left(e^{\frac{2\pi\ell}{k} i}, 0\right) \in \mathbb{S}^1 \times \mathbb{R}^{n-2}$
- the remainder term ϕ is invariant under the action of a suitable group of symmetries



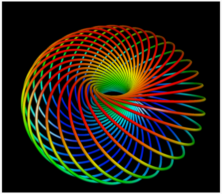
$\mathbb{R}^4$ has a richer topology than $\mathbb{R}^3$: Hopf fibration

- $\mathbb{S}^3 \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$ and $\mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$
- $h : \mathbb{S}^3 \to \mathbb{S}^2$, $h(z_1, z_2) := (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$ is the Hopf map
- Each fiber over a point of $\mathbb{S}^2$ is a great circle in $\mathbb{S}^3$
- Fibers over different points are linked great circles in $\mathbb{S}^3$, i.e. Hopf link
- Each fiber over a circle $\mathbb{S}^1 \subset \mathbb{S}^2$ is a torus in $\mathbb{S}^3$, i.e. Clifford torus





Hopf link



Clifford Torus

Concentration on q great circles which are linked ($n \geq 4$) Del Pino, Musso, Pacard & Pistoia (2013)

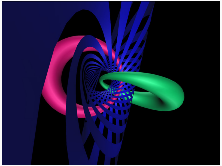
- $\mathbb{S}^n \subset \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3}$
- $\Lambda_0 := \left\{ \frac{1}{\sqrt{2}}(z, z, \hat{0}) : \ z \in S^1 \right\}$ is a great circle of $\mathbb{S}^n$
- for any $q \geq 1$ let $t_q : \mathbb{S}^n \to \mathbb{S}^n$ be $t_q(z_1, z_2, x^*) = \left(e^{-\frac{i\pi}{q}} z_1, e^{\frac{i\pi}{q}} z_2, x^* \right)$
- $\Lambda := \Lambda_0 \cup t_q \Lambda_0 \cup \cdots \cup t_q^{q-1} \Lambda_0$ is the union of q great circles
- Any two such great circles are linked and correspond to a Hopf link

There exists $k_0 > 0$ such that for any $k \geq k_0$ there exists a u_k solution to

$$-\Delta_{\mathfrak{g}_0} u + \frac{n(n-2)}{4} u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{S}^n$$

such that as $k \to \infty$

- $u_k \to c_n$ uniformly on compact sets of $\mathbb{S}^n \setminus \Lambda$
- u_k blow-up negatively at the $2k \times q$ points in Λ



2 linked great circles

If $n \geq 4$ then the problem

$$-\Delta u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{R}^n$$

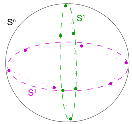
admits infinitely many sign-changing non-radial solutions which are invariant under the action of $\mathcal{D}_k \times \mathcal{D}_h \times \mathcal{O}(n-3)$ for k and h large enough.

The solutions look like

$$u(y) = U(y) - \sum_{\ell=1}^{k} \delta^{-\frac{n-2}{2}} U\left(\frac{y-\xi_\ell}{\delta}\right) - \sum_{j=1}^{k} \delta^{-\frac{n-2}{2}} U\left(\frac{y-\eta_j}{\epsilon}\right) + \phi(y)$$

where

- $U(y) = \alpha_n \frac{1}{(1+|y|^2)^{\frac{n-2}{2}}}$ solves $-\Delta U = U^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$
- the concentration parameters $\delta \sim \frac{d_n}{k^2}$
- the concentration points $\xi_\ell \sim \left(e^{\frac{2\pi\ell}{k}i}, 0, 0\right) \in \mathbb{S}^1 \times \mathbb{R}^2 \times \mathbb{R}^{n-4}$
- the concentration points $\eta_j \sim \left(0, 0, e^{\frac{2\pi j}{h}i}\right) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^{n-4}$
- the remainder term ϕ is invariant under the action of a suitable group of symmetries



Concentration on Clifford torus ($n \geq 5$) Del Pino, Musso, Pacard & Pistoia (2013)

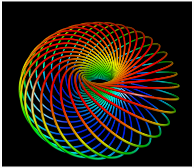
- $\mathbb{S}^n \subset \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3}$
- $\mathbb{T}^2 := \frac{1}{\sqrt{2}} \left(\mathbb{S}^1 \times \mathbb{S}^1 \right) \times \{0\}$ is a Clifford torus of $\mathbb{S}^n$

There exists $k_0 > 0$ such that for any $k \geq k_0$ there exists u_k solution to

$$-\Delta_{g_0} u + \frac{n(n-2)}{4} u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{S}^n$$

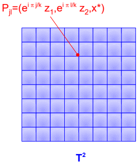
such that and as $k \to \infty$

- $u_k \to c_n$ uniformly on compact sets of $\mathbb{S}^n \setminus \mathbb{T}^2$
- u_k blow-up negatively at the $(2k)^2$ points of $\mathbb{T}^2$



$\mathbb{T}^2$: Clifford Torus

On top of $2k$ linked great circles we put $2k$ negative bubbles



Let us go back to the system...

...and remind

Whenever there exists a of sign-changing solutions to the single equation

$$-\Delta w = |w|^{p-1} w \text{ in } \mathbb{R}^n$$

hopefully (using the same strategy) one could try to find a positive solution to the system

$$(S) \quad \begin{cases} -\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}} \quad \text{in } \mathbb{R}^n, \quad i = 1, \ldots, m \\ u_1, \ldots, u_m \in D^{1,2}(\mathbb{R}^n) \end{cases}$$

in the competitive case, i.e. $\beta_{ij} < 0$

Ding's result

Ding found (via variational tools) infinitely many sign-changing symmetric solutions to

$$-\Delta u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{R}^n$$

Competitive systems Clapp & Pistoia (2018)

In the competitive case, i.e. $\beta < 0$, the system

$$(S_\beta) \qquad \begin{cases} -\Delta u = \mu_1 u^p + \beta u^{\frac{p-1}{2}} v^{\frac{p+1}{2}} \text{ in } \mathbb{R}^n, \\ -\Delta v = \mu_2 v^p + \beta u^{\frac{p+1}{2}} v^{\frac{p-1}{2}} \text{ in } \mathbb{R}^n, \\ u, v \in D^{1,2}(\mathbb{R}^n), \end{cases}$$

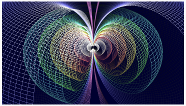
has a positive (symmetric) solution (u_β, v_β).
Moreover, it is a non-syncronized solution when $\beta \to -\infty$, since a phase separation phenomenon occurs.

More precisely:

- $u_\beta \to u_\infty$ and $v_\beta \to v_\infty$ strongly in $D^{1,2}(\mathbb{R}^n)$ as $\beta \to -\infty$
- The function $w := u_\infty - v_\infty$ is a sign-changing solution to

$$-\Delta w = \mu_1 \left(w^+\right)^p - \mu_2 \left(w^-\right)^p \text{ in } \mathbb{R}^n \quad \Rightarrow \quad \boxed{u_\infty v_\infty \equiv 0 \text{ in } \mathbb{R}^n}$$

- $\{u_\infty > 0\} \cong \mathbb{S}^{k-1} \times \mathbb{B}^{n+1-k}$ and $\{v_\infty > 0\} \cup \{\infty\} \cong \mathbb{B}^k \times \mathbb{S}^{n-k}$ are a partition of $\mathbb{R}^n$



The level sets are tori
$\{u_\infty = v_\infty = 0\}$ is the green one

Some comments

Our results concerns competitive systems with only two components!
Clapp and Szulkin (2019), Clapp, Saldaña and Szulkin (2019) extended the result to systems with an arbitrary number of components in a fully competitive regime, i.e. all the β_{ij}'s are negative.

The proof relies on a variational argument similar to the one used by Ding when he shows the existence of sign-changing solutions to the single equation.

The proof: The variational setting

- Let $\mathbf{D} := D^{1,2}(\mathbb{R}^n) \times D^{1,2}(\mathbb{R}^n)$. The solutions to the system (S_β) are the critical points of the $\mathcal{C}^1$-functional $E : \mathbf{D} \to \mathbb{R}$

$$E(u,v) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + |\nabla v|^2\right) - \frac{1}{2^*}\int_{\mathbb{R}^n} \left(\mu_1 |u|^{2^*} + \mu_2 |v|^{2^*} + 2\beta |u|^{\frac{2^*}{2}} |v|^{\frac{2^*}{2}}\right).$$

- The fully non-trivial solutions lie on the set

$$\mathcal{N} := \{(u,v) \in \mathbf{D} : u \neq 0,\ v \neq 0,\ \partial_u E(u,v)u = 0,\ \partial_v E(u,v)v = 0\},$$

  called the Nehari manifold. $\mathcal{N}$ has the following properties:
  - $\mathcal{N}$ is a closed $\mathcal{C}^1$-submanifold of codimension 2 of $\mathbf{D}$.
  - It is a natural constraint for E, i.e. a critical point of the restriction of E to $\mathcal{N}$ is a critical point of E.
  - Every (u,v) in $\mathcal{N}$ is fully non-trivial.
- $\mathbf{D} \hookrightarrow L^{2^*}(\mathbb{R}^n) \times L^{2^*}(\mathbb{R}^n)$ is not compact $\Rightarrow \inf_{(u,v)\in\mathcal{N}} E(u,v)$ is not attained!

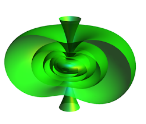
The proof: How to recover the compactness

- Let $\Gamma = \mathcal{O}(k) \times \mathcal{O}(n+1-k) \subset \mathcal{O}(n+1)$ for $2 \leq k \leq n-1$.
- Γ acts isometrically on the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.
- Γ acts conformally on $\mathbb{R}^n$ via the stereographic projection $\pi : \mathbb{S}^n \setminus \{S\} \to \mathbb{R}^n$.
- A function $u \in D^{1,2}(\mathbb{R}^n)$ is Γ-invariant if $su = u$. Set

$$\mathbf{D}^\Gamma := \{(u, v) \in \mathbf{D} \ : \ u \text{ and } v \text{ are } \Gamma\text{-invariant}\}.$$

- The Γ-orbit $\Gamma x := \{sx : s \in \Gamma\} \cong \mathbb{S}^{k-1} \times \mathbb{S}^{n-k}$ or $\cong \mathbb{S}^{k-1}$ or $\cong \mathbb{S}^{n-k}$.
- $1 \leq \dim(\Gamma x) \leq n-1 \ \Rightarrow \ \mathbf{D}^\Gamma \hookrightarrow L^{2^*}(\mathbb{R}^n) \times L^{2^*}(\mathbb{R}^n)$ is compact
- $E : \mathcal{N}^\Gamma \to \mathbb{R}$ satisfies the Palais-Smale condition, where

$$\mathcal{N}^\Gamma := \{(u, v) \in \mathcal{N} \ : \ u \text{ and } v \text{ are } \Gamma\text{-invariant}\}$$



The level sets are tori

- The critical points of $E : \mathcal{N}^\Gamma \to \mathbb{R}$ are Γ-invariant solutions to the system (S_β).
- E has infinitely many critical points and a positive minimizer on $\mathcal{N}^\Gamma$
  (via a refined Ljusternik-Schnirelmann category result due to Szulkin (1988))
- The system (S_β) has infinitely many $\Gamma-$invariant solutions and a least energy positive solution.

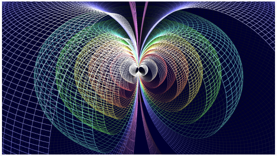
The proof: Phase separation

For $\beta \to -\infty$, let $(u_\beta, v_\beta) \in \mathcal{N}_\beta^\Gamma$ satisfy $u_\beta \geq 0$, $v_\beta \geq 0$ and $E_\beta(u_\beta, v_\beta) = \min_{\mathcal{N}_\beta^\Gamma} E_\beta$.

Then, after passing to a subsequence,

- $u_\beta \to u_\infty$ and $v_\beta \to v_\infty$ strongly in $D^{1,2}(\mathbb{R}^n)^\Gamma$,
-
$$E_\beta(u_\beta, v_\beta) \text{ is bounded } \Rightarrow \underbrace{\beta}_{\substack{\downarrow \\ -\infty}} \underbrace{\int_{\mathbb{R}^n} |u_\beta|^{\frac{2^*}{2}} |v_\beta|^{\frac{2^*}{2}}}_{\substack{\downarrow \\ 0}} \text{ is bounded } \Rightarrow \boxed{u_\infty \cdot v_\infty \equiv 0 \text{ in } \mathbb{R}^n}$$
- $w := u_\infty - v_\infty$ is a $\Gamma-$invariant least energy sign-changing solution to the single equation
$$-\Delta w = \mu_1 (w^+)^p - \mu_2 (w^-)^p \text{ in } \mathbb{R}^n.$$

- The domains
$\{y \in \mathbb{R}^n \ : \ u_\infty(y) > 0\} \cong \mathbb{S}^{k-1} \times \mathbb{B}^{n+1-k}$
and
$\{y \in \mathbb{R}^n \ : \ v_\infty(y) > 0\} \cup \{\infty\} \cong \mathbb{B}^k \times \mathbb{S}^{n-k}$
are a Γ-invariant partition of $\mathbb{R}^n$
- The interface
$\{u_\infty = v_\infty = 0\} \cong \mathbb{S}^{k-1} \times \mathbb{S}^{n-k}$



The level sets are tori

Del Pino, Musso, Pacard & Pistoia's result

We built (via a Ljapunov-Schmidt procedure) infinitely many sign-changing solutions to

$$-\Delta u = |u|^{\frac{4}{n-2}} u \text{ in } \mathbb{R}^n$$

which are the superposition of one positive bubble and a large number of negative bubbles which blow-up at points which in turn are regularly arranged along one or more (linked) great circles of $\mathbb{R}^n$.

Competitive systems in 3D Guo, Li & Wei (2014)

In the competitive case, i.e. $\beta < 0$, the system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u = u^5 + \beta u^2 v^3 \text{ in } \mathbb{R}^3, \\ -\Delta v = v^5 + \beta u^3 v^2 \text{ in } \mathbb{R}^3 \end{cases}$$

admits infinitely many positive non-radial and non-syncronized solutions.

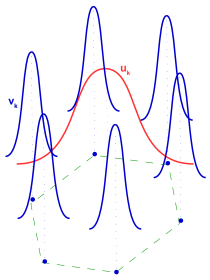
More precisely, there exists $k_0 > 0$ such that for any $k \geq k_0$ there exists a solution (u_k, v_k) such that

$$u_k(y) = U(y) + \phi(y)$$

$$v_k(y) = \sum_{\ell=1}^{k} \frac{1}{\sqrt{\delta}} U\left(\frac{y - \xi_\ell}{\delta}\right) + \psi(y)$$

where

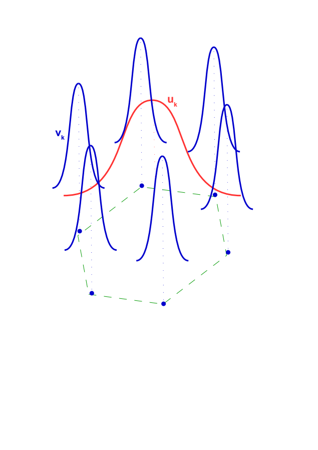
- $U(y) = \alpha_3 \frac{1}{\left(1+|y|^2\right)^{\frac{1}{2}}}$ solves $-\Delta U = U^5$ in $\mathbb{R}^3$
- the concentration parameter $\delta = \delta(k) \to 0$
- the concentration points $\xi_\ell \sim \left(e^{\frac{2\pi\ell}{k} i}, 0\right) \in \mathbb{S}^1 \times \mathbb{R}$
- the remainder terms ϕ, ψ are invariant under the action of a suitable group of symmetries



Look!

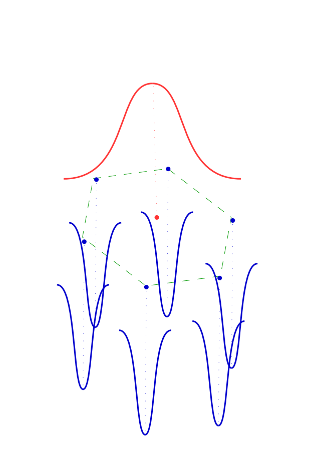
Guo, Li & Wei constructed positive solutions to the competitive system, i.e. $\beta < 0$

$$\begin{cases} -\Delta u = u^5 + \beta u^2 v^3 \text{ in } \mathbb{R}^3, \\ -\Delta v = v^5 + \beta u^3 v^2 \text{ in } \mathbb{R}^3 \end{cases}$$



Del Pino, Musso, Pacard and Pistoia constructed sign-changing solutions to the equation

$$-\Delta w = w^5 \text{ in } \mathbb{R}^3$$



A couple of questions

Q1: is it possible to find solutions to systems in higher dimensions $n \geq 4$?

Q2: is it possible to find solutions to systems with more than 2 components?

Partial positive answers: Chen, Medina & Pistoia (2023)

- Q1: Yes!
- Q2: Yes!

Our results

A system with 2 components in $\mathbb{R}^4$ Chen, Medina & Pistoia (2021)

If $\beta < 0$ small enough the system

$$\begin{cases} -\Delta u = u^3 + \beta u v^2 \text{ in } \mathbb{R}^4, \\ -\Delta v = v^3 + \beta v u^2 \text{ in } \mathbb{R}^4 \end{cases}$$

admits an arbitrary large number of non-radial and non-syncronized solutions.

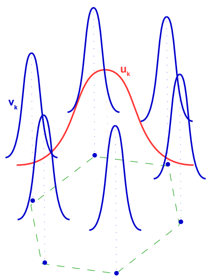
More precisely, for any even integer $k \geq 2$ there exists $\beta_k < 0$ such that for any $\beta \in (\beta_k, 0)$ there exists a solution

$$u(y) = U(y) + \phi(y)$$

$$v(y) = \sum_{\ell=1}^{k} \frac{1}{\delta} U\left(\frac{y - \xi_\ell}{\delta}\right) + \psi(y)$$

where

- $U(y) = \alpha_4 \frac{1}{1+|y|^2}$ solves $-\Delta U = U^3$ in $\mathbb{R}^4$
- the concentration parameter $\delta \sim e^{-\frac{d}{|\beta|}}$ as $|\beta| \to 0$ (for some $d > 0$)
- the k concentration points $\xi_1, \ldots \xi_k$ belong to a great circle $\mathbb{S}_1$ and are the vertices of a regular polygon
- the remainder terms ϕ, ψ are invariant under the action of a suitable group of symmetries



A system with $q+1$ components in $\mathbb{R}^4$ Chen, Medina & Pistoia (2021)

For any $\alpha \in \mathbb{R}$, if $\beta < 0$ small enough the system

$$\begin{cases} -\Delta u = u^3 + \beta u \sum_{j=1}^{q} v_j^2 \text{ in } \mathbb{R}^4 \\ -\Delta v_i = v_i^3 + \beta v_i u^2 + \alpha v_i \sum_{j \neq i}^{q} v_j^2 \text{ in } \mathbb{R}^4, \ i = 1, \dots, q \end{cases}$$

admits an arbitrary large number of non-radial and non-syncronized solutions.

A system with $q+1$ components in $\mathbb{R}^4$ Chen, Medina & Pistoia (2021)

For any $\alpha \in \mathbb{R}$, if $\beta < 0$ small enough the system

$$\begin{cases} -\Delta u = u^3 + \beta u \sum_{j=1}^{q} v_j^2 \text{ in } \mathbb{R}^4 \\ -\Delta v_i = v_i^3 + \beta v_i u^2 + \alpha v_i \sum_{j \neq i}^{q} v_j^2 \text{ in } \mathbb{R}^4, \ i = 1, \dots, q \end{cases}$$

admits an arbitrary large number of non-radial and non-syncronized solutions.

More precisely, for any even integer $k \geq 2$ there exists $\beta_k < 0$ such that for any $\beta \in (\beta_k, 0)$ there exists a solution

- $u(y) = U(y) + \phi$, $U(y) = \alpha_4 \frac{1}{1+|y|^2}$ solves $-\Delta U = U^3$ in $\mathbb{R}^4$
- $v_i(x) = v(\mathscr{S}_i x)$, if $i = 1, \dots, q$, $v(y) = \sum_{\ell=1}^{k} \frac{1}{\delta} U\left(\frac{y-\xi_\ell}{\delta}\right) + \psi$
- $\mathscr{S}_i := \begin{pmatrix} \mathcal{R}\left(\frac{2(i-1)\pi}{qk}\right) & 0 \\ 0 & \mathcal{R}\left(-\frac{2(i-1)\pi}{qk}\right) \end{pmatrix}$, $\mathscr{S}_1 = \texttt{Identity}$ where $\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,
- the concentration parameter $\delta \sim e^{-\frac{d}{|\beta|}}$ as $|\beta| \to 0$ (for some $d > 0$)
- the k concentration points $\xi_1, \dots \xi_k$ belong to a great circle $\mathbb{S}_1$ and are the vertices of a regular polygon
- the remainder terms ϕ, ψ are invariant under the action of a suitable group of symmetries

If $q = 1$ ...

the system

$$\begin{cases} -\Delta u = u^3 + \beta u v^2 \text{ in } \mathbb{R}^4, \\ -\Delta v = v^3 + \beta v u^2 \text{ in } \mathbb{R}^4 \end{cases}$$

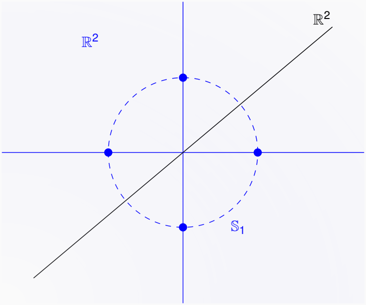
has a solution (u, v) such that

$$u \sim U$$

and

v blows-up at k points regularly arranged long the great circle $\mathbb{S}_1 := \{x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\}$

The peaks of v



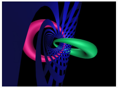
If $q \geq 2$ ...

the system

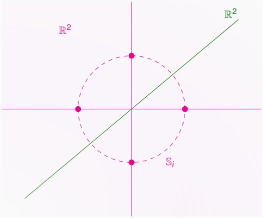
$$\begin{cases} -\Delta u = u^3 + \beta u \sum_{j=1}^{q} v_j^2 \text{ in } \mathbb{R}^4 \\ -\Delta v_i = v_i^3 + \beta v_i u^2 + \alpha v_i \sum_{j \neq i}^{q} v_j^2 \text{ in } \mathbb{R}^4, \; i = 1, \dots, q \end{cases}$$

has a solution $(u, v_1, \dots, v_q)$ such that

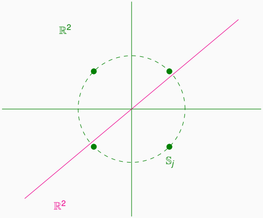
- $u \sim U$
- each component v_i blows-up at k points regularly arranged along the great circle $\mathbb{S}_i$
- $\mathbb{S}_i$ with $\mathbb{S}_j$ $(i \neq j)$ is an Hopf link.



The peaks of v_i



The peaks of v_j



The proof

Reducing the system to a non-local system via symmetries

$$\begin{cases} -\Delta u = u^3 + \beta u \sum_{j=1}^{q} v_j^2 \text{ in } \mathbb{R}^4 \\ -\Delta v_i = v_i^3 + \beta v_i u^2 + \alpha v_i \sum_{j \neq i}^{q} v_j^2 \text{ in } \mathbb{R}^4, \; i = 1, \dots, q \end{cases}$$

We look for a symmetric solution

$$\boxed{u(x) = u(\mathscr{S}_i x)} \quad \text{and} \quad \boxed{v_i(x) = v(\mathscr{S}_i x)} \quad \text{if } i = 1, \dots, q$$

where

$$\mathscr{S}_i := \begin{pmatrix} \mathcal{R}\left(\frac{2(i-1)\pi}{qk}\right) & 0 \\ 0 & \mathcal{R}\left(-\frac{2(i-1)\pi}{qk}\right) \end{pmatrix}, \; \mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where (u, v) solves the non-local system (with only 2 equations)

$$\begin{cases} -\Delta u = u^3 + \underbrace{\beta u \sum_{j=1}^{q} v^2(\mathscr{S}_j x)}_{\text{non-local term}} \text{ in } \mathbb{R}^4 \\ -\Delta v = v^3 + \beta v u^2 + \underbrace{\alpha v \sum_{j=2}^{q} v^2(\mathscr{S}_j x)}_{\text{non-local term}} \text{ in } \mathbb{R}^4, \end{cases}$$

Finding a solution to the non-local system

$$\begin{cases} -\Delta u = u^3 + \beta u \sum_{j=1}^{q} v^2(\mathscr{S}_j x) \text{ in } \mathbb{R}^4 \\ -\Delta v = v^3 + \beta v u^2 + \alpha v \sum_{j=2}^{q} v^2(\mathscr{S}_j x) \text{ in } \mathbb{R}^4, \end{cases}$$

Using a Ljapunov-Schmidt procedure, for any even integer k we build a solution

$$u(x) = U(x) + \phi \quad \text{and} \quad v(x) = \sum_{\ell=1}^{k} \frac{1}{\delta} U\left(\frac{x-\xi_\ell}{\delta}\right) + \psi$$

where

- the concentration parameter $\delta \sim e^{-\frac{d}{|\beta|}}$ for some $d > 0$ as $\beta \to 0$
- the k concentration points $\xi_1, \dots, \xi_k \in \mathbb{S}_1 := \left\{x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\right\}$ and

$$\xi_\ell \sim \sqrt{2}\left(\cos\frac{2(\ell-1)\pi}{k}, \sin\frac{2(\ell-1)\pi}{k}, \cos\frac{2(\ell-1)\pi}{k}, \sin\frac{2(\ell-1)\pi}{k}\right) \text{ as } \beta \to 0$$

- the remainder terms ϕ, ψ are invariant under the action of a suitable group of symmetries

Some remarks

$$-\Delta u_i = \beta_{ii} u_i^p + \sum_{j \neq i} \beta_{ij} \underbrace{u_i^{\frac{p-1}{2}} u_j^{\frac{p+1}{2}}}_{\text{coupling term}} \quad \text{in } \mathbb{R}^n,$$

We can build solutions using the Ljapunov-Schmidt procedure only when $n = 3$ or $n = 4$.

- $n = 3 \Rightarrow p = 5 \Rightarrow u_j^3 u_i^2$ is superlinear in both u_j and u_i
  Guo, Li & Wei constructed infinitely many solutions for any coupling parameter $\beta < 0$ using the large number k of peaks as a parameter.
- $n = 4 \Rightarrow p = 3 \Rightarrow u_j^2 u_i$ is superlinear in u_j and linear in u_i
  This is an obstacle to the contraction property that is needed in the fixed point theory. This is why we need to take β_{ij}'s as a small parameter and to fix the number k of bubbles. However, we strongly believe that this assumption is due to technical reasons!
- $n \geq 5 \Rightarrow p < 3 \Rightarrow u_j^{\frac{p+1}{2}} u_i^{\frac{p-1}{2}}$ is sublinear in both u_j and u_i
  The linearized problem becomes singular and new ideas are needed!

Open problems

What happens in higher dimension, i.e. $n \geq 5$?

What happens if $\beta < 0$ is not small and we take as a parameter the large number of peaks?
Work in progress with Antonio Fernandez & Maria Medina.

Can we find this kind of solutions in a more general setting, without assuming the full symmetry?

A final remark

- **Pinwheel** solutions, i.e. *solutions which have the property that each component is obtained from the previous one by a rotation,*
  have been found (using variational arguments) for competitive **subcritical** (i.e. $1 < p < \frac{n+2}{n-2}$) systems in the presence of an external radial trapping potential by Clapp & Pistoia (2023)

$$-\Delta u_i + V(x) u_i = u_i^p + \beta \sum_{j \neq i} u_i^{\frac{p-1}{2}} u_j^{\frac{p+1}{2}} \quad \text{in } \mathbb{R}^n,$$

- The critical case $p = \frac{n+2}{n-2}$ is open!

THANK YOU FOR YOUR ATTENTION