Exponential PDEs in high dimensions

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Log-determinants on 2−manifolds

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Application: compactness of isospectral domains/surfaces:

- B. Osgood, R. Phillips, P. Sarnak, JFA '88
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- conformal Laplacian $L_g = -\Delta_g + \frac{(n-2)}{4(n-1)} R_g$
- Paneitz operator $P_g = \Delta_g^2 - \mathrm{div}(\frac{2}{3} R_g g - 2Ric_g) \circ \nabla$
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$$\Rightarrow \gamma_2(L_g) = 6\gamma_3(L_g), \quad \gamma_2(\not{D}_g^2) = \frac{132}{7}\gamma_3(\not{D}_g^2)$$

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Gauss-Bonnet formula: $4\pi^2 \chi(M) = \int_M \left(\frac{|W_g|_g^2}{8} + Q_g\right) dv_g$, where $\chi(M)$ Euler characteristic of M

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[M. Gursky, CMP '99] $k_{L_g} < 32\pi^2$ if $R_g \geq 0$ except on $(\mathbb{S}^4, g_0)$

Main results

In general $k_{A_g} \leq 8\pi^2(-\gamma_2)$ fails (products of negatively-curved surfaces, hyperbolic manifolds)

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$\frac{\gamma_2}{\gamma_3} \geq 6$, w_n blow-up sequence of $\mathcal{N}_g(w_n) + U_g = \mu_n e^{4w_n}$ in M
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We find conformal metrics with $U = const.$ as saddle points of F_{A_g}:

Theorem 2 (P.E., A. Malchiodi, JDG '24)

$\frac{\gamma_2}{\gamma_3} \geq 6$, M *compact manifold s.t.* $k_{A_g} \notin 8\pi^2(-\gamma_2)\mathbb{N}$
Then $\exists\ \tilde{g} \in [g]$ *with* $U_{\tilde{g}} = const.$

On the standard sphere

- T. Branson, A. Chang, P. Yang, CMP '92 ["unique" maximizer for $F_{L_{g_0}}$ and $F_{\not{D}^2_{g_0}}$]
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Via stereographic projection $F'_{L_{g_0}}(w) = 0$ on $(\mathbb{S}^4, g_0)$ equivalent to

$$3\Delta^2 W + 2\Delta(|\nabla W|^2) - 4\mathrm{div}\,[(\Delta W + |\nabla W|^2)\nabla W] = 16\pi^2 e^{4W}$$

in $\mathbb{R}^4$ with $W \sim -2\log|x|$ at infinity.

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Question: $\int_{\mathbb{R}^4} e^{4W} < +\infty$ does it implies $W \sim -2\log|x|$ at ∞?

Quasilinear PDEs in dimension n

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Arises also in

$$-\Delta_{n,g} u + |\nabla u|_g^{n-2} \mathrm{Ric}_g(\nabla u, \nabla u) = |\nabla u|_{\hat{g}}^{n-2} \mathrm{Ric}_{\hat{g}}(\nabla u, \nabla u) e^{nu}$$

see

- S. Ma, J. Qing, Calc. Var '21 & Adv. Math. '22

The n-Liouville equation

Consider the euclidean quasilinear PDE

$$-\Delta_n u = V e^u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \qquad (P)$$

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$\Delta_n u = \text{div}(|\nabla u|^{n-2}\nabla u)$ $n-$Laplace operator, $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and

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- H. Brézis, F. Merle, Comm. PDE '91
- Y.Y. Li, I. Shafrir, Indiana Univ. Math. J. '94
- Y.Y. Li, Comm. Math. Phys. '99
- C.C. Chen, C.S. Lin, Comm. Pure Appl. Math. '02 & '03

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Notation: $c_n = n(\frac{n^2}{n-1})^{n-1}$

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Aim: classify solutions of

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Theorem 5 (P.E., AIHP '18)

Any solution U of $(P)_\infty$ has the form $U_{\lambda,p}$. In particular $\int_{\mathbb{R}^n} e^U = c_n \omega_n$

The semilinear case $n = 2$

Classification known since a long ago, proved in different ways:

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An alternative approach: via Pohozev identity in

- P.-L. Lions, Appl. Anal. '81
- S. Kesavan, F. Pacella, Appl. Anal. '94
- S. Chanillo, M. Kiessling, Geom. Funct. Anal. '95

Quantization and classification issues

If U is a solution of $(P)_\infty \Rightarrow$ the Kelvin transform $\hat{U}$ satisfies

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Mass quantization for singular $n-$Liouville equation:

$$-\Delta_n U = e^U - \gamma \delta_0 \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^U < +\infty$$

- P. E., Calc. Var. PDE '21 [if $n \geq 2$]
- J. Prajapat, G. Tarantello, Proc. Edinburgh '01 [if $n = 2$]

Interior blow-up

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In the two-dimensional case $\alpha_p \in 8\pi\mathbb{N}$ is shown in

- Y.Y. Li, I. Shafrir, Indiana Univ. Math. J. '94

based on a Harnack inequality of sup + inf type

- I. Shafrir, C.R.A.S. '92

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Theorem 6 (P.E., M. Lucia, preprint)

Given $K \subset \Omega$ compact and $C_1 < \frac{1}{n-1}$, there exists $C_2 > 0$ so that

$$C_1 \max_K u_k + \inf_{\Omega} u_k \leq C_2$$

About c_1

The constant c_1 is not explicit but $0 < c_1 \leq (n\omega_n)^{-\frac{1}{n-1}}$, see

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$$\Rightarrow \quad u(x) - \inf_{\Omega} u \geq -\frac{1}{2\pi} \int_{B_\delta(x)} \log \frac{|z-x|}{\delta} f(z) dz$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\delta t \log \frac{t}{\delta} f(t\theta + x) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\delta \frac{dt}{t} \int_0^t r f(r\theta + x) dr = \frac{1}{2\pi} \int_0^\delta [\int_{B_t(x)} f] \frac{dt}{t}$$

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- since $V_k e^{u_k} \sim V(p) e^{U_{\mu_k^{-1}, x_k}}$ in $B_{R\mu_k}(x_k)$ with $p = \lim_{k \to +\infty} x_k$, further compare w_k from below with the radial case where $c_1 = (n\omega_n)^{-\frac{1}{n-1}}$

Quantization for mass concentration

By $\sup + \inf$ inequalities one gets decay estimates on $V_k e^{u_k}$:

$$V_k e^{u_k} \le C \frac{\mu_k^\alpha}{|x - x_k|^{n+\alpha}} \qquad \text{in } B_{\frac{d_k}{2}}(x_k) \setminus B_{R\mu_k}(x_k)$$

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Following clusters by clusters, rather standard to show that

Theorem 7 (P.E., M. Lucia, preprint)

$$\alpha_p \in c_n \omega_n \mathbb{N}$$

extending the two-dimensional result in

- Y.Y. Li, I. Shafrir, Indiana Univ. Math. J. '94

Optimal sup + inf inequality

The decay exponent α is in general $\alpha < \frac{n}{n-1}$. When blow-up is simple, is it possible to reach $\alpha = \frac{n}{n-1}$? Equivalent to

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The answer is related to the following fundamental expansion

$$u_k - U_{\mu_k^{-1}, x_k} = O(1) \qquad \text{in } B_\delta(p)$$

and optimal constant $C_1 = \frac{1}{n-1}$ in the sup + inf inequality, see

- D. Bartolucci, C.C. Chen, C.S. Lin, G. Tarantello, Comm. PDE '04
- H. Brézis, Y.Y. Li, I. Shafrir, JFA '93
- Y.Y. Li, Comm. Math. Phys. '99

Work in progress

In collaboration with M. Lucia: optimal $\sup + \inf$ inequality (i.e. with $C_1 = \frac{1}{n-1}$) when $\sup_{k \in \mathbb{N}} \operatorname{osc}_{\partial B_\delta(x_k)} u_k < +\infty$

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Crucial property: $-\Delta_n u_k + \Delta_n v_k = V_k e^{u_k} - \fint_{\partial B_r(x_k)} V_k e^{u_k}$

Open questions

- optimal $\sup + \inf$ inequality without boundary control
- more precise asymptotic expansion of u_k towards the case $\lambda \in c_n\omega_n\mathbb{N}$
- blow-up solutions as $\lambda \to c_n\omega_n\mathbb{N}$. Via Lyapunov-Schmidt reduction: Pistoia, Premoselli, Vétois, ...
- multiplicity results via Morse theory

Thanks for your attention