Existence and asymptotics of solutions for nonlinear elliptic equations with a gradient-dependent nonlinearity [1]

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Introduction

The singularity problem for quasi-linear elliptic eqns of the general form

$$\mathrm{div}\, \mathbf{A}(x, u, \nabla u) = B(x, u, \nabla u). \tag{1}$$

For a domain Ω in $\mathbb{R}^N$ with $0 \in \Omega$ and a soln u in a suitable sense (e.g., in $\mathcal{D}'(\Omega \setminus \{0\})$), the following questions are of interest:

Can u be extended to the whole domain Ω in a natural way so that the new function satisfies the equation in Ω (a removable singularity)? Otherwise, what is the behaviour of u near 0?

The topic of isolated singularities has received much attention in connection with geometry (minimal surfaces), the Yamabe problem and mathematical physics.

Serrin's pioneering papers (Acta Math, 1964 & 1965): For a domain Ω in $\mathbb{R}^N$ with $0 \in \Omega$, assume that $\mathbf{A}(x,u,\xi)$ and $B(x,u,\xi)$ are, respectively, vector and scalar measurable functions in $\Omega \times \mathbb{R} \times \mathbb{R}^N$ satisfying:

$$\begin{cases} |\mathbf{A}(x,u,\xi)| \leq \beta_0|\xi|^{p-1} + \beta_1|u|^{p-1} + \beta_2, \\ \xi \cdot \mathbf{A}(x,u,\xi) \geq |\xi|^p - \beta_3|u|^p - \beta_4, \\ |B(x,u,\xi)| \leq \beta_6|\xi|^{p-1} + \beta_3|u|^{p-1} + \beta_5, \end{cases} \tag{2}$$

$\forall (x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, where $1 < p \leq N$ is fixed, $\beta_0 > 0$ is a constant and β_i $(1 \leq i \leq 6)$ are measurable functions on Ω belonging to suitable Lebesgue classes: $\beta_1, \beta_2 \in L^{N/(p-1-\varepsilon)}$, $\beta_6 \in L^{N/(1-\varepsilon)}$ and $\beta_j \in L^{N/(p-\varepsilon)}$ for $j = 3,4,5$, where $\varepsilon > 0$. Then for any positive soln u of (1), we have:

1. u can be extended as a continuous soln of (1) in Ω;
2. or $\exists c_1, c_2 > 0$ s.t. $c_1 \leq u(x)/\mu(x) \leq c_2$ in a neighbourhood of 0, where μ denotes the fundamental soln of the p-harmonic eqn $-\mathrm{div}\,(|\nabla\mu|^{p-2}\nabla\mu) = \delta_0$ (Dirac mass at 0) in $\mathcal{D}'(\mathbb{R}^N)$.

Serrin's papers [23, 24] have generated much research on isolated singularities in the attempt to find analogues results for other nonlinear PDEs. For the development of the singularity theory to nonlinear second-order diff. eqns of elliptic (parabolic) type, see Véron's books [31] (1996) and [32] (2017).

The challenge remains to address the singularity problem for quasi-linear elliptic eqns in divergence form such as (1) when the growth of B is bigger than that of $\mathbf{A}$. In this case, a crucial difficulty lies in that solns with singularities stronger than that of μ may appear.

I. Quasilinear elliptic equations

Let B_1 denote the open unit ball in $\mathbb{R}^N$ ($N \geq 2$) centred at 0. For $1 < p \leq N$ and $q > 0$, the profile near 0 of all positive solns for

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u \quad \text{in } B^* := B_1 \setminus \{0\} \tag{3}$$

depends on the position of q w.r.t. $p-1$ as well as $q_* = \frac{N(p-1)}{N-p}$.

- Serrin [23, 24] (Acta Math. 1964, 1965): If $0 < q \leq p-1$, then
  1. u can be extended as a continuous soln of (3) in B_1;
  2. or $\exists c_1, c_2 > 0$ s.t. $c_1 \leq u(x)/\mu(x) \leq c_2$ in a neighbourhood of 0.

- Vázquez–Véron [27] (Manuscripta Math., 1980/1981):

If $q \geq q_*$ (for $1 < p < N$), then any positive soln can be extended as a continuous soln of (3) in B_1 (removability).

• Friedman–Véron [15] (Arch. Ration. Mech. Anal., 1986):
If $p-1<q<q_*$, then as $|x|\to 0$, exactly one of the following holds:
(i_1) u can be extended as a continuous soln of (3) in B_1;
(i_2) $\exists\lambda\in(0,\infty)$ s.t. $u(x)/\mu(x)\to\lambda$ (weak singularity) and

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)+|u|^{q-1}u=\lambda^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_1).$$

(i_3) $|x|^{p/(q+1-p)}u(x)\to\gamma_{N,p,q}$ (strong singularity), where

$$\gamma_{N,p,q}:=\left[\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{pq}{q+1-p}-N\right)\right]^{1/(q+1-p)}.$$

Generalizations: C.–Du [10] (JFA, 2010) and Chang–C. [5] (AIHP 2017) generalized the results of Friedman–Véron [15] and Vázquez–Véron [27] to nonlinear elliptic equations in divergence form

$$\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u)=b(x)\,h(u) \quad \text{in } B^*:=B_1\setminus\{0\}. \tag{4}$$

II: Gradient-dependent nonlinearities and Hardy potentials

We study the existence of positive solutions and the classification of their behaviour near zero for nonlinear elliptic equations of the form

$$\mathbb{L}_{\rho,\lambda}(u) := \Delta u + (2 - N - 2\rho)\,\frac{x \cdot \nabla u}{|x|^2} + \frac{\lambda}{|x|^2}\,u = |x|^{\theta} u^q \,|\nabla u|^m \tag{5}$$

in $\Omega \setminus \{0\}$, where Ω is an open subset Ω_0 of $\mathbb{R}^N$ containing zero.
Assumption: Let ρ, λ, θ, m, and q be real parameters such that

$$m > 0, \quad q \geq 0, \quad \text{and} \quad \kappa := m + q - 1 > 0. \tag{6}$$

The solutions of (5) are understood in the sense of distributions: a non-negative function $u \in C^1(\Omega \setminus \{0\})$ that satisfies

$$\begin{aligned} &-\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + (2 - N - 2\rho) \int_{\Omega} \frac{x \cdot \nabla u}{|x|^2} \varphi \, dx + \int_{\Omega} \frac{\lambda\, u\varphi}{|x|^2} \, dx \\ &= \int_{\Omega} |x|^{\theta} u^q |\nabla u|^m \varphi \, dx \quad \forall\, \varphi \in C_c^1(\Omega \setminus \{0\}). \end{aligned} \tag{7}$$

Background.

A. Without the gradient factor: The case $\rho = (2-N)/2$ and $m = 0$ is understood very well:

- C. [8] (Mem. AMS, 2014) for $\lambda \leq (N-2)^2/4$ and $\theta > -2$.
- Wei–Du [33] (JDE, 2017) for $\lambda > (N-2)^2/4$ and $\theta > -2$;
- C.–Farcăşeanu (JDE, 2021) for $\lambda, \theta \in \mathbb{R}$ and $\Omega = \Omega_0$ or Ω_∞ or $\mathbb{R}^N$.

B. Including the gradient factor, but no Hardy potential:
Ching–C. [6] (Analysis & PDE, 2015) studied the case $\rho = (2-N)/2$, $\lambda = \theta = 0$ and $m \in (0, 2)$.

Recall that $\kappa = m + q - 1 > 0$. For every $\rho, \lambda, \theta \in \mathbb{R}$, we define

$$\Theta := \frac{\theta + 2 - m}{\kappa} \quad \text{and} \quad \ell = \ell(\rho, \lambda, \Theta) := \Theta^2 + 2\rho\Theta + \lambda. \tag{8}$$

- When $\lambda \leq \rho^2$, we define $\Theta_\pm$ as the roots of $t^2 + 2\rho t + \lambda = 0$:

$$\Theta_\pm := -\rho \pm \sqrt{\rho^2 - \lambda}.$$

We have $\ell \leq 0$ if and only $\lambda \leq \rho^2$ and $\Theta_- \leq \Theta \leq \Theta_+$.

The behaviour near zero of the positive solutions of (5) is closely linked with two special solutions $\Phi^\pm_{\rho,\lambda}$ of $\mathbb{L}_{\rho,\lambda}(\cdot) = 0$ defined by

$$\Phi^-_{\rho,\lambda}(x) = |x|^{-\Theta_-}, \quad \Phi^+_{\rho,\lambda}(x) = \begin{cases} |x|^{-\Theta_+} & \text{if } \lambda < \rho^2 \\ |x|^{-\Theta_+} \log \frac{1}{|x|} & \text{if } \lambda = \rho^2 \end{cases}$$

We use a modified Kelvin transform for a positive soln u of (5) as follows

$$K[u](x) := u(\widetilde{x}), \quad \text{where } \widetilde{x} = x/|x|^2 \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}. \tag{9}$$

We observe that $|\nabla K[u](x)|^2 = |\nabla u(\widetilde{x})|^2 |\widetilde{x}|^2$ and

$$\operatorname{div}\left(|x|^{2(2-N)} \nabla K[u](x)\right) = |\widetilde{x}|^{2N} \Delta u(\widetilde{x}) \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}. \tag{10}$$

Hence, the modified Kelvin transform $K[u]$ of u satisfies

$$\Delta K[u] + (2 - N + 2\rho) \frac{x \cdot \nabla K[u]}{|x|^2} + \frac{\lambda}{|x|^2} K[u] = |x|^{-\theta + 2m - 4} (K[u])^q |\nabla K[u]|^m \tag{11}$$

for every $x \in \mathbb{R}^N \setminus \{0\}$. If u solves (5) with $\Omega = \mathbb{R}^N$, then $K[u]$ solves an equation of the same type as (5) except that

$$\rho \text{ in } (5) \longmapsto -\rho \text{ in } (11),$$
$$\theta \text{ in } (5) \longmapsto -\theta + 2m - 4 \text{ in } (11).$$

For $\beta > 0$ and $\mu \neq 0$, we define $F_{\beta,\mu}(r) = \beta^{\frac{1}{\kappa}} |\mu|^{-\mu} |\log r|^{\mu}$ for $r \in (0,1)$.

Table: Important asymptotic profiles near zero

Condition	**Relevant asymptotic profile**
$\lambda \leq \rho^2$	$\Phi^{-}_{\rho,\lambda}(x) = \|x\|^{-\Theta_-}$
$\lambda \leq \rho^2$	$\Phi^{+}_{\rho,\lambda}(x) = \begin{cases} \|x\|^{-\Theta_+} & \text{if } \lambda < \rho^2 \\ \|x\|^{-\Theta_+} \log \frac{1}{\|x\|} & \text{if } \lambda = \rho^2 \end{cases}$
$\ell > 0$, $\Theta \neq 0$	$U_{\ell,\Theta}(x) = (\|\Theta\|^{-m}\ell)^{1/\kappa} \|x\|^{-\Theta}$
$\ell > 0$, $\Theta = 0$	$U_{\ell,0}(x) = F_{\lambda,\mu}(\|x\|)$, $\mu = m/\kappa$
$\lambda \leq \rho^2$, $\Theta = \Theta_- \neq 0$	$U_{0,\Theta_-}(x) = \|\Theta_-\|^{-\frac{m}{\kappa}} \|x\|^{-\Theta_-} F_{\beta,\mu}(\|x\|)$ $\beta = 2\sqrt{\rho^2 - \lambda}$, $\mu = -\frac{1}{\kappa}$ if $\Theta_- \neq -\rho$ $\beta = 1 - \frac{1}{\mu}$, $\mu = -\frac{2}{\kappa}$ if $\Theta_- = -\rho$
$\lambda = \Theta = 0$, $\rho\,(m-1) > 0$	$F_{2\|\rho\|,\mu}(\|x\|)$, $\mu = (m-1)/\kappa$
$\lambda = \Theta = \rho = 0$, $m \in (0,2)$	$F_{1-\frac{1}{\mu},\mu}(\|x\|)$ and $\mu = (m-2)/\kappa$

Theorem 1 (C.-Fărcășeanu, preprint)

Let (6) *hold,* $\lambda < 0$ *and* $\Theta, \rho \in \mathbb{R}$. *Let* u *be any positive soln of* (5).

(I) *If* $\Theta < \Theta_-$, *then it holds*

$$u(x) \sim U_{\ell,\Theta}(x) \text{ as } |x| \to 0 \text{ i.e., } \lim_{|x|\to 0} |x|^{\Theta} u(x) = (\ell/|\Theta|^m)^{1/\kappa}. \tag{12}$$

(II) *If* $\Theta = \Theta_-$, *then we have*

$$u(x) \sim U_{0,\Theta_-}(x) \text{ as } |x| \to 0. \tag{13}$$

(III) *If* $\Theta_- < \Theta \leq \Theta_+$, *then the following limit exists*

$$\lim_{|x|\to 0} u(x)/\Phi^-_{\rho,\lambda}(x) \in (0,\infty). \tag{14}$$

(IV) *If* $\Theta > \Theta_+$, *then exactly one of the following situations occurs:*
(A) (12) *holds;* (B) *There exists* $\lim_{|x|\to 0} u(x)/\Phi^+_{\rho,\lambda}(x) \in (0,\infty)$.
(C) (14) *holds.*

Theorem 2 (C.-Fărcășeanu, preprint)

Let (6) *hold and* $\rho, \theta, \lambda \in \mathbb{R}$ *such that* $\ell > 0$. *Let* u *be any given positive solution of* (5) *with* $\Omega = \Omega_0$. *If* $\lambda \geq 0$ *and* $\Theta < 0$, *we assume that* $\lim_{|x|\to 0} u(x) = 0$. *If* $\lambda \leq \rho^2$ *and* $\Theta > \Theta_+$, *we also assume that*

$$\lim_{|x|\to 0} u(x)/\Phi^+_{\rho,\lambda}(x) = \infty. \qquad (15)$$

Then, $u(x) \sim U_{\ell,\Theta}(x)$ *as* $|x| \to 0$, *where*

$$U_{\ell,\Theta}(x) = \begin{cases} (|\Theta|^{-m}\ell)^{1/\kappa}\,|x|^{-\Theta} & \text{if } \Theta \neq 0, \\ \lambda^{\frac{1}{\kappa}}(m/\kappa)^{-m/\kappa}|\log|x||^{m/\kappa} & \text{if } \Theta = 0. \end{cases} \qquad (16)$$

Theorem 3 (Existence)

Let (6) *hold,* $\Theta \neq 0$ *and* $\ell > 0$. *Then,* $\forall R > 0$, *Eq.* (5) *in* $B_R(0) \setminus \{0\}$ *has infinitely many positive solns satisfying* $u(x) \sim U_{\ell,\Theta}(x)$ *as* $|x| \to 0$.

Theorem 4 (Refined asymptotics)

Let (6) *hold,* $\Theta \neq 0$ *and* $\ell > 0$. *Let* $u > 0$ *be a radial soln of* (5) *in* $B_R(0) \setminus \{0\}$ *for some* $R > 0$ *s.t.* $u(x) \sim U_{\ell,\Theta}(x)$ *as* $|x| \to 0$. *Assume that* $u \not\equiv U_{\ell,\Theta}$ *in any interval* $(0, r_*)$ *with* $r_* \in (0, R)$. *Then,* $\exists \mu_0 \in \mathbb{R} \setminus \{0\}$ *s.t.*

$$\begin{aligned} &\lim_{r \to 0^+} r^{-\xi_0} \left(\frac{r u'(r)}{u(r)} + \Theta \right) = \mu_0, \\ &u(r) = U_{\ell,\Theta}(r) \left(1 + \frac{\mu_0}{\xi_0} r^{\xi_0} (1 + o(1)) \right) \text{ as } r \to 0^+, \end{aligned} \tag{17}$$

where ξ_0 *is the* positive *root of the following quadratic equation (in* ξ*)*

$$\xi^2 + \left(\frac{\ell\, m}{\Theta} - 2 (\rho + \Theta) \right) \xi - \ell \kappa = 0. \tag{18}$$

Theorem 5

Let (6) hold, $\Theta = 0$, $\lambda > 0$, $\rho \in \mathbb{R}$. Assume $u \in C^2(B_R(0) \setminus \{0\})$ is a positive radial soln of (5) in $B_R(0) \setminus \{0\}$ s.t. $u(x) \sim U_{\ell,0}(x)$ as $|x| \to 0$.

(a) *Assume that either $\rho \neq 0$ or $q \neq 1$. For $r > 0$ small, we define*

$$t = \frac{u^2(r)}{r^2(u'(r))^2}, \quad X(t) = t\left(t^{-\frac{m}{2}} u^\kappa(r) - \lambda\right) - 2\rho\sqrt{t} + \frac{q-1}{m}. \qquad (19)$$

(a_1) *Then, as $r \to 0^+$, we have $t \to \infty$ and $X(t) \to 0$. Moreover, as $r \to 0^+$*

$$u(r) = \begin{cases} U_{\ell,0}(r)\left(1 + \dfrac{2\rho\, m}{\lambda\kappa^2}\, \dfrac{\log\log\frac{1}{r}}{\log\frac{1}{r}}(1+o(1))\right) & \text{if } \rho \neq 0, \\[2ex] U_{\ell,0}(r)\left(1 + \dfrac{(q-1)\, m}{\lambda\kappa^3}\, \dfrac{(1+o(1))}{\left(\log\frac{1}{r}\right)^2}\right) & \text{if } \rho = 0,\ q \neq 1. \end{cases} \qquad (20)$$

Theorem 6 (Continuation)

(a_2) *There exists $T_0 > 0$ large s.t. $|X(t)| < \kappa/(2m)$ $\forall t \geq T_0$ and*

$$\begin{aligned} \frac{dX}{dt} = & \frac{\lambda\, m\, X(t)}{2\left(\frac{\kappa}{m} - X(t)\right)} + \left(1 + \frac{m\, X(t)}{\frac{\kappa}{m} - X(t)}\right) \frac{\rho}{\sqrt{t}} \\ & + \left(1 + \frac{m\, X(t)}{2\left(\frac{\kappa}{m} - X(t)\right)}\right) \frac{\frac{1-q}{m} + X(t)}{t}. \end{aligned} \tag{21}$$

(b) *If $\rho = 0$ and $q = 1$, then there exists a constant $c \in \mathbb{R}$ such that*

$$u(r) = U_{\ell,0}(r) + c = \lambda^{\frac{1}{\kappa}} \, |\log r| + c \quad \textit{for all } r > 0 \textit{ small.} \tag{22}$$

Characterization of solutions modeled by U_{0,Θ_-}:

Theorem 7 (C.-Fărcășeanu, preprint)

Let (6) *hold,* $\lambda \leq \rho^2$ *and* $\Theta = \Theta_- \neq 0$. *Let* u *be any given positive solution of* (5) *with* $\Omega = \Omega_0$. *If* $\lambda \geq 0$ *and* $\Theta < 0$, *we further assume that* $\lim_{|x|\to 0} u(x) = 0$. *Then,* u *satisfies* (13): $u(x) \sim U_{0,\Theta_-}(x)$ *as* $|x| \to 0$.

In Theorem 8 with $\lambda < \rho^2$ we obtain a sharp existence result, confirming that in any of the cases of Theorem 7, there exist infinitely many positive radial sols of (5) in $B_R(0) \setminus \{0\}$ s.t. (13) holds. We set

$$\mathfrak{p} := \frac{|\Theta_-|^m}{2\sqrt{\rho^2 - \lambda}} > 0 \quad \text{and} \quad \mathfrak{M} := \frac{\kappa + 1}{2\sqrt{\rho^2 - \lambda}} + \frac{m}{\Theta_-}. \tag{23}$$

For every $r \in (0,1)$, we define $U_{0,\Theta_-}(r)$ by

$$U_{0,\Theta_-}(r) := v_0\, r^{-\Theta_-} |\log r|^{-\frac{1}{\kappa}} \text{ with } v_0 := (\mathfrak{p}\kappa)^{-\frac{1}{\kappa}}. \tag{24}$$

Theorem 8

Let (6) *hold,* $\rho \in \mathbb{R}$, $\lambda < \rho^2$ *and* $\Theta = \Theta_- \neq 0$.

(a) *Let* (13) *hold for a positive radial soln* u *of* (5) *in* $B_R(0) \setminus \{0\}$.

(a_1) *If* $\mathfrak{M} \neq 0$, *then we have*

$$u(r) = U_{0,\Theta_-}(r)\left[1 + \frac{2\,\mathfrak{M}}{\kappa^2}\,\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}(1+o(1))\right] \text{ as } r \to 0^+. \quad (25)$$

(a_2) *If* $\mathfrak{M} = 0$ *and* $m \neq 1$, *then*

$$u(r) = U_{0,\Theta_-}(r)\left[1 + \frac{m(m-1)}{\kappa^3\,(\Theta_-)^2}\,\frac{(1+o(1))}{\left(\log\frac{1}{r}\right)^2}\right] \text{ as } r \to 0^+. \quad (26)$$

(a_3) *If* $\mathfrak{M} = 0$ *and* $m = 1$, *then there exists a constant* $C \in \mathbb{R}$ *such that*

$$u(r) = U_{0,\Theta_-}(r)\left(1 + C|\log r|^{-1}\right)^{-\frac{1}{\kappa}} \quad \text{for every } r > 0 \text{ small.} \quad (27)$$

(b) *For every* $R \in (0,\infty)$, *equation* (5) *in* $B_R(0) \setminus \{0\}$ *has infinitely many positive radial solutions satisfying* (13).

Lemma 9

Let (6) *hold,* $\rho \in \mathbb{R}$, $\lambda < \rho^2$ *and* $\Theta = \Theta_- \neq 0$. *Let* $u > 0$ *be a radial soln of* (5) *in* $B_R(0) \setminus \{0\}$ *s.t.* (13) *holds. We define*

$$t = (r^{\Theta_-} u(r))^{-\kappa} \quad \text{and} \quad X(t) = t\left(ru'(r)/u(r) + \Theta_-\right) - \mathfrak{p}. \tag{28}$$

(a) $\exists T_0 > 4\,\mathfrak{p}/|\Theta_-|$ *large s.t.* $|X(t)| \leq \mathfrak{p}/2\ \forall t \geq T_0$ *and*

$$\begin{aligned} \frac{dX}{dt} =& \frac{\sqrt{\rho^2 - \lambda}}{\kappa\,(X(t) + \mathfrak{p})}\,X(t) + \left(1 + \frac{1}{\kappa}\right)\frac{X(t) + \mathfrak{p}}{t} \\ &+ \frac{|\Theta_-|^m}{\kappa\,(X(t) + \mathfrak{p})}\left[1 - \left(1 - \frac{X(t) + \mathfrak{p}}{t\,\Theta_-}\right)^m\right]. \end{aligned} \tag{29}$$

(b) *If* $\mathfrak{M} \neq 0$, *then* $tX(t) \to -2\mathfrak{p}^2\mathfrak{M}$ *as* $t \to \infty$ *and* (25) *holds.*

(c) *If* $\mathfrak{M} = 0$ *and* $m \neq 1$, *then* $\lim_{t\to\infty} t^2 X(t) = m(m-1)\mathfrak{p}^3/(\Theta_-)^2$ *and* (26) *holds.*

(d) *If* $\mathfrak{M} = 0$ *and* $m = 1$, *then* $X \equiv 0$ *on* $[T_0, \infty)$ *and* (27) *holds.*

When $\lambda = \rho^2 \neq 0$ and $\Theta = \Theta_\pm = -\rho$, we recall that U_{0,Θ_-} is defined by

$$U_{0,\Theta_-}(r) := \left(2\,(\kappa+2)\,\kappa^{-2}\,|\Theta_-|^{-m}\right)^{\frac{1}{\kappa}} r^{-\Theta_-}|\log r|^{-\frac{2}{\kappa}} \text{ for } r \in (0,1). \quad (30)$$

Theorem 10

Let (6) *hold,* $\lambda = \rho^2 \neq 0$ *and* $\Theta = \Theta_\pm = -\rho$.

(a) *If* (13) *holds for a positive radial soln* u *of* (5) *in* $B_R(0) \setminus \{0\}$, *then*

$$u(r) = U_{0,\Theta_-}(r)\left[1 - \frac{4m(2+\kappa)}{\rho\kappa^2(3\kappa+4)}\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}(1+o(1))\right] \text{ as } r \to 0^+. \quad (31)$$

(b) *For every* $R \in (0,\infty)$, *equation* (5) *in* $B_R(0) \setminus \{0\}$ *has infinitely many positive radial solutions satisfying* (31).

We define $\mathfrak{Q}$ as follows

$$\mathfrak{Q} := |\rho|^{\frac{m}{2}} / \sqrt{1 + \kappa/2}. \qquad (32)$$

Lemma 11

Let (6) hold, $\lambda = \rho^2 \neq 0$ and $\Theta = \Theta_\pm = -\rho$. Let $u > 0$ be a positive radial soln of (5) in $B_R(0) \setminus \{0\}$ s.t. (13) holds. For $r > 0$ small, we define

$$e^t = (r^{-\rho} u(r))^{-\frac{\kappa}{2}} \quad \text{and} \quad X(t) = e^t \left(r u'(r)/u(r) - \rho \right) - \mathfrak{Q}. \qquad (33)$$

(a) *As $r \to 0^+$, we have $e^t / \log(1/r) \to \mathfrak{Q}\kappa/2$ and $X(t) \to 0$.*

(b) *$\exists T_0 > \log(4\mathfrak{Q}/|\rho|)$ large s.t. $|X(t)| \leq \mathfrak{Q}/2$ for all $t \geq T_0$ and*

$$\frac{dX}{dt} = \left(1 + \frac{2}{\kappa}\right)(X + \mathfrak{Q}) \left\{ 1 - \frac{\mathfrak{Q}^2}{(X + \mathfrak{Q})^2} \left[1 + \frac{e^{-t}}{\rho}(X + \mathfrak{Q}) \right]^m \right\}. \qquad (34)$$

(c) *We have $e^t X(t) \to \frac{2m|\rho|^m}{\rho(3\kappa+4)}$ as $t \to \infty$ and (31) holds.*

Theorem 12 (C.-Fărcășeanu, preprint)

Let (6) *hold and* $\Theta = \lambda = 0$. *Let* u *be any positive soln of* (5).

(a) *If* $m \in (0,1)$ *and* $\rho < 0$, *then the following dichotomy occurs:*

(i) *Either* $\lim_{|x|\to 0} u(x) = 0$ *and, more precisely,*

$$u(x) \sim F_{2|\rho|,\mu}(x) \text{ as } |x| \to 0, \quad \text{where } \mu = (m-1)/\kappa. \tag{35}$$

(ii) *Or* $0 < \liminf_{|x|\to 0} u(x) \leq \limsup_{|x|\to 0} u(x) < \infty$.

(b) *If* $m \geq 1$ *and* $\rho < 0$, *then the alternative in (a)(ii) always holds.*

(c) *If* $m > 1$ *and* $\rho > 0$, *then the following trichotomy occurs:*

(i) *Either* $\lim_{|x|\to 0} u(x) = +\infty$ *and, more precisely,* (35) *holds.*

(ii) *Or* $0 < \liminf_{|x|\to 0} u(x) \leq \limsup_{|x|\to 0} u(x) < \infty$.

(iii) *Or* $\lim_{|x|\to 0} u(x) = 0$ *and*
$0 < \liminf_{|x|\to 0} u(x)/\Phi^-_{\rho,\lambda}(x) \leq \limsup_{|x|\to 0} u(x)/\Phi^-_{\rho,\lambda}(x) < \infty$.

(d) *If* $0 < m \leq 1$ *and* $\rho > 0$, *then either (c)(ii) or (c)(iii) holds.*

Theorem 13 (C.-Fărcășeanu, preprint)

Let (6) *hold,* $\Theta = \lambda = \rho = 0$. *Let* u *be a positive soln of* (5).

(a) *If* $m \in (0,2)$, *then the following dichotomy occurs:*

(i) *Either* $\lim_{|x|\to 0} u(x) = 0$ *and, more precisely,*

$$\lim_{|x|\to 0} \frac{u(x)}{F_{1-\frac{1}{\mu},\mu}(x)} = 1, \quad \text{where } \mu = \frac{m-2}{\kappa}. \tag{36}$$

(ii) *Or there exists* $\lim_{|x|\to 0} u(x) \in (0,\infty)$.

(b) *If* $m \geq 2$, *then there always exists* $\lim_{|x|\to 0} u(x) \in (0,\infty)$.

For classification: Gradient estimates via Bernstein technique (for $\lambda \leq 0$).

Proposition 1

Let $m \geq 0$, $q \geq 0$, $m + q > 1$, $\lambda \leq 0$, and $\theta \in \mathbb{R}$. Then, there exists a positive constant C_1, depending only on m, N, q, λ and θ, such that for every positive solution u of (5) and $r_0 > 0$ with $\overline{B_{2r_0}(0)} \subset \Omega$, we have

$$|\nabla u(x)| \leq C_1 \frac{u(x)}{|x|} \quad \text{for all } \ 0 < |x| \leq r_0. \tag{37}$$

Idea of the proof. If $\lambda = 0$, then the conclusion follows from Lemma 2.1 in Ching–C. [Proc. Roy. Soc. Edinburgh Sect. A (2020)]. For $\lambda < 0$ we proceed similarly, but modifications appear in the latter part of the proof.

Comparison Principles

Idea of the Proof. Careful constructions of super-solutions and comparison principles.

Lemma 14 (Comparison principle, I)

Let D be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Let $\widehat{B}(x, z, \boldsymbol{\xi}) : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous in $D \times \mathbb{R} \times \mathbb{R}^N$ and continuously differentiable with respect to $\boldsymbol{\xi}$ for $|\boldsymbol{\xi}| > 0$ in $\mathbb{R}^N$. Assume that $\widehat{B}(x, z, \boldsymbol{\xi})$ is non-decreasing in z for fixed $(x, \boldsymbol{\xi}) \in D \times \mathbb{R}^N$. Let u_1 and u_2 be non-negative $C^1(D) \cap C(\overline{D})$ (distributional) solutions of

$$\begin{cases} \Delta u_1 - \widehat{B}(x, u_1, \nabla u_1) \geq 0 & \text{in } D, \\ \Delta u_2 - \widehat{B}(x, u_2, \nabla u_2) \leq 0 & \text{in } D. \end{cases} \tag{38}$$

Suppose $|\nabla u_1| + |\nabla u_2| > 0$ in D. If $u_1 \leq u_2$ on ∂D, then $u_1 \leq u_2$ in D.

Lemma 15 (Comparison principle, II)

Let D be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Assume that $\widehat{B}(x, z, \boldsymbol{\xi}) : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $D \times \mathbb{R} \times \mathbb{R}^N$ and is non-decreasing in z for fixed $(x, \boldsymbol{\xi}) \in D \times \mathbb{R}^N$. Let u_1 and u_2 be (distribution) solutions in $W^{1,\infty}_{\mathrm{loc}}(D)$ of (38). If

$$u_1 \leq u_2 + M \quad \text{on } \partial D,$$

where M is a positive constant, then $u_1 \leq u_2 + M$ in D.

Lemma 14 follows from Pucci–Serrin (JDE, 2004) [20, Theorem 10.1]. The second comparison principle in Lemma 15 is included in Corollary 3.5.2 to Theorem 3.5.1 in [21], where the class $C^1(D)$ is weakened to $W^{1,\infty}_{\mathrm{loc}}(D)$. Theorem 3.5.1 in Pucci–Serrin (2007) [21], like Theorem 10.1 in [20], is essentially Theorem 10.7(i) in Gilbarg and Trudinger's book [16] with the exception that the functions $\widehat{A}$ and $\widehat{B}$ in [20, 21] are allowed to be singular at $\boldsymbol{\xi} = 0$. This is then compensated in [20, Theorem 10.1] by the additional condition that $|\nabla u_1| + |\nabla u_2| > 0$ in D.

Aim: To obtain existence of positive (radial) solutions in each of the cases prescribed by our sharp classification results.

We use a dynamical systems approach. We illustrate it for the existence of positive solutions of (5) modeled by U_0 near zero.

Assume that $\Theta \neq 0$ and $\ell > 0$, where ℓ is given by (8). We define M_0 by

$$M_0 := (|\Theta|^{-m}\ell)^{\frac{1}{\kappa}} > 0. \tag{39}$$

We prove that for every $R > 0$, there exist infinitely many positive radial solutions $u(x) = u(|x|)$ of (5) in $B_R(0)$ satisfying

$$\lim_{r \to 0^+} \frac{u(r)}{r^{-\Theta}} = \lim_{r \to 0^+} \frac{u'(r)}{-\Theta\, r^{-\Theta-1}} = M_0. \tag{40}$$

Fix $R > 0$. We first assume that u is a positive radial solution of (5) in $B_R(0) \setminus \{0\}$ satisfying (40). Then, there exists $r_0 \in (0, R)$ small such that for every $r \in (0, r_0]$, we have

$$|r^{\Theta} u(r) - M_0| \leq \frac{M_0}{2} \quad \text{and} \quad |r^{\Theta+1} u'(r) + \Theta\, M_0| \leq \frac{|\Theta|\, M_0}{2}. \tag{41}$$

For every $r \in (0, r_0]$, we set

$$t = \log(r_0/r)$$

and define $(X_1(t), X_2(t), X_3(t))$ as follows

$$\begin{cases} X_1(t) = r^{\Theta}\, u(r) - M_0, \\ X_2(t) = r^{\Theta+1}\, u'(r) + \Theta\, M_0, \\ X_3(t) = r. \end{cases} \tag{42}$$

By the assumption in (40), it is clear that

$$(X_1(t), X_2(t), X_3(t)) \to (0, 0, 0) \quad \text{as } t \to \infty.$$

Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be smooth functions such that

$$f_1(s) = (s + M_0)^q \quad \text{for every } |s| \leq M_0/2,$$
$$f_2(s) = (\Theta M_0 - s)^m \text{ if } \Theta > 0 \text{ and } f_2(s) = (s - \Theta M_0)^m \text{ if } \Theta < 0$$

for every $|s| \leq |\Theta|\, M_0/2$. From our choice of r_0 so that (41) holds, we have

$$f_1(X_1(t)) = (X_1(t) + M_0)^q \quad \text{and} \quad f_2(X_2(t)) = |X_2(t) - \Theta\, M_0|^m \text{ for all } t \geq 0.$$

Then, $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t))$ is a solution of the following autonomous first order differential system: for all $t \geq 0$

$$\begin{cases} X_1'(t) = -\Theta\, X_1(t) - X_2(t) := H_1(\mathbf{X}), \\ X_2'(t) = \lambda\, X_1(t) - (\Theta + 2\rho)\, X_2(t) - f_1(X_1(t))\, f_2(X_2(t)) + \ell\, M_0 := H_2(\mathbf{X}), \\ X_3'(t) = -X_3(t) := H_3(\mathbf{X}). \end{cases} \tag{43}$$

Hence, we have obtained a nonlinear differential system of the form

$$\mathbf{X}'(t) = \mathbf{H}(\mathbf{X}) \tag{44}$$

where $\mathbf{H} = (H_1, H_2, H_3)$ is specified in (43). Remark that in our situation, X_3 in (42) is a positive solution of $X_3'(t) = -X_3(t)$ for $t \geq 0$.

We next describe the behaviour of (43) near its critical point $\mathbf{0} = (0, 0, 0)$. Since $\mathbf{0}$ is a *hyperbolic* critical point, the behaviour of the nonlinear system (43) near $\mathbf{X} = \mathbf{0}$ is approximated by the behaviour of its linearization $\mathbf{X}' = A\mathbf{X}$ at $\mathbf{X} = \mathbf{0}$. The matrix A, representing $D\mathbf{H}(\mathbf{0})$, is given by

$$A = \begin{bmatrix} -\Theta & -1 & 0 \\ \lambda - \ell q & -\Theta - 2\rho + \frac{\ell m}{\Theta} & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (45)$$

The matrix A has no zero or pure imaginary eigenvalues so that $\mathbf{0}$ is a *hyperbolic* critical point. More precisely, $\mathbf{0}$ is a *saddle point* since two eigenvalues of A are negative and the other is positive. The eigenvalues of A are $\mu_3 = -1$ and the roots $\mu_{1,2}$ of the quadratic equation

$$\mu^2 + 2\psi\mu - \ell\, k = 0, \quad \text{where } \psi := \Theta + \rho - \frac{\ell m}{2\,\Theta}. \quad (46)$$

Since $\mu_1\mu_2 = -\ell k < 0$, the equation in (46) has one positive root μ_2 and one negative root μ_1 given by

$$\mu_1 := -\psi - \sqrt{\psi^2 + \ell k}, \quad \mu_2 := -\psi + \sqrt{\psi^2 + \ell k}.$$

For the unstable eigenvalue μ_2, an associated eigenvector is $v_2 = (1, \rho - \frac{\ell m}{2\Theta} - \sqrt{\psi^2 + \ell k}, 0)$. The stable eigenvalue μ_1 (and μ_3, respectively) has an eigenvector $v_1 = (1, \rho - \frac{\ell m}{2\Theta} + \sqrt{\psi^2 + \ell k}, 0)$ (and $v_3 = (0, 0, 1)$, respectively).

The stable subspace E^s of the linear system $\mathbf{X}'(t) = A\mathbf{X}$ at $\mathbf{0}$ is the plane spanned by v_1 and v_3, that is,

$$\left(\rho - \frac{\ell m}{2\Theta} + \sqrt{\psi^2 + \ell k}\right) x_1 - x_2 = 0.$$

Let ϕ_t be the flow of the nonlinear system (43). By the Stable Manifold Theory, there exists a two-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system $\mathbf{X}'(t) = A\mathbf{X}$ at $\mathbf{0}$ such that for all $t \geq 0$, we have $\phi_t(S) \subseteq S$ and for all $\mathbf{x}_0 \in S$,

$$\lim_{t\to\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}.$$

The stable manifold S is local since it is only defined in a small neighbourhood of $\mathbf{0}$. Moreover, since our function $\mathbf{H}$ in (44) is of class C^∞ on a small open neighbourhood E of $\mathbf{0}$, it is known that the stable manifold S is of class C^∞ as well. The invariant stable manifold S is (locally) unique.

Thank you for your attention!

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