Singular solutions to Hitchin's equation and harmonic maps to the conformal 3-sphere

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Moduli spaces and singularities

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Higgs bundles

Hitchin's equation

Let Σ be a Riemann surface and E a complex vector of rank 2 together with a trivialization of $\Lambda^2 E$. Let

- $\bar{\partial}_E$ a holomorphic structure on E inducing $\Lambda^2 E \cong \mathcal{O}$
- $\Phi \in \Omega^{1,0}(\Sigma, \mathfrak{sl}(E))$ a traceless Higgs field

Definition

$(E, \bar{\partial}_E, \Phi)$ is called $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle $:\Leftrightarrow \bar{\partial}_E \Phi = 0$.

If $(E, \bar{\partial}_E, \Phi)$ ist stable, $\exists!$ hermitian metric h inducing trivial flat metric on $\Lambda^2 E \cong \mathcal{O}$ and solving Hitchin's equation

$$F_h + [\Phi \wedge \Phi^{*_h}] = 0$$

Hence

$$\mathcal{M}_{Higgs} \cong \{(A, \phi) : F_A + [\phi \wedge \phi^*] = 0, \bar{\partial}_A = 0\}/\mathcal{G}$$

Higgs bundles

Equivariant harmonic maps

If (A, ϕ) solution to Hitchin's equation w.r.t. background hermitian metric $h \Rightarrow$

$$\nabla = d_A + \phi + \phi^*$$

is a flat $\mathrm{SL}(2, \mathbb{C})$-connection. Let $\rho : \pi_1 \Sigma \to \mathrm{SL}(2, \mathbb{C})$ be the monodromy representation.

Integrating ∇ yields a ρ-equivariant harmonic map

$$f : \tilde{\Sigma} \to \mathbb{H}^3 = \{A \in \mathrm{SL}(2, \mathbb{C}) : A^* = A, A > 0\}$$

as follows: Let F be a parallel frame field on $\tilde{\Sigma}$, i.e. $\nabla F = 0$. Then

$$F(p) : (\mathbb{C}^2, \langle \cdot, \cdot \rangle) \xrightarrow{\cong} (E_p, h_p)$$

and

$$F(p)^* : (E_p, h_p) \xrightarrow{\cong} (\mathbb{C}^2, \langle \cdot, \cdot \rangle)$$

to obtain $f = F^* F$.

Harmonic maps to the conformal 3-sphere

Lightcone model

Consider $\mathbb{R}^{4,1}$ with the standard Minkowski inner product

$$\langle \cdot\,, \cdot \rangle = -dx_0^2 + dx_1^2 + \ldots + dx_4^2$$

and the lightcone

$$\mathcal{L} = \{x \in \mathbb{R}^{4,1} \mid \langle x, x \rangle = 0\}.$$

The map

$$\mathbb{R}^4 \to P\mathbb{R}^{4,1}, \quad (x_1, x_2, x_3, x_4) \mapsto [1 : x_1 : x_2 : x_3 : x_4]$$

restricts to a diffeomorphism

$$\mathbb{S}^3 \to P\mathcal{L} \subset P\mathbb{R}^{4,1}$$

between the 3-sphere and the projectivized light cone.

Harmonic maps to the conformal 3-sphere

Conformal structure

If σ is a (local) section of $\pi\colon \mathcal{L} \to P\mathcal{L}$ then the conformal structure is represented by the metric

$$g_\sigma(X, Y) = \langle d\sigma(X), d\sigma(Y)\rangle.$$

Obviously, two lifts give conformally equivalent metrics.

- The spherical lift

  $$\sigma_{sph} : P\mathcal{L} \to \mathcal{L}, \quad [x] \mapsto x/x_0$$

  yields the round metric g_{sph} on $\mathbb{S}^3$.
- The hyperbolic lift

  $$\sigma_{hyp} : \{[x] \in P\mathcal{L} \mid x_1 \neq 0\} \to \mathcal{L}, \quad [x] \mapsto x/x_1$$

  yields the hyperbolic metric g_{hyp} on $\mathbb{H}^3_+ \cup \mathbb{H}^3_-$.

Harmonic maps to the conformal 3-sphere

2-sphere at infinity

We obtain decomposition

$$\mathbb{S}^3 = \mathbb{H}^3_+ \cup \mathbb{S}^2_\infty \cup \mathbb{H}^3_-$$

where

$$\begin{aligned}
\mathbb{H}^3_+ &= \{[x] \in P\mathcal{L} \mid x_1 = 1, x_0 > 0\}, \\
\mathbb{S}^2_\infty &= \{[x] \in P\mathcal{L} \mid x_1 = 0\}, \\
\mathbb{H}^3_- &= \{[x] \in P\mathcal{L} \mid x_1 = 1, x_0 < 0\}.
\end{aligned}$$

We call $\mathbb{S}^2_\infty$ the 2-sphere at infinity.

- $\mathrm{SO}(4,1) = \mathrm{Conf}_+(\mathbb{S}^3)$
- $\mathrm{Isom}_+(\mathbb{H}^3) = \mathrm{SO}(3,1)_0$ acts conformally on $\mathbb{S}^3$ preserving $\mathbb{S}^2_\infty$ and the two components $\mathbb{H}^3_\pm$

Harmonic maps to the conformal 3-sphere

Conformality at infinity

Let Σ be a Riemann surface. For $f:\Sigma \to \mathbb{S}^3 = P\mathcal{L}$ smooth let

$$f_{hyp} = \sigma_{hyp} \circ f : \Sigma \setminus f^{-1}(\mathbb{S}^2_\infty) \to \mathcal{L}$$

the hyperbolic lift of f and

$$q_{hyp} = \langle \partial f_{hyp}, \partial f_{hyp} \rangle \in \Gamma(\Sigma \setminus f^{-1}(\mathbb{S}^2_\infty), K^2_\Sigma)$$

the hyperbolic Hopf differential.

Definition

f conformal at infinity $:\Leftrightarrow$ q_{hyp} extends to a smooth section in $\Gamma(\Sigma, K^2_\Sigma)$.

Remark

If f is conformal at infinity and $f_\sigma = \sigma \circ f$ any smooth lift, then $q_\sigma = \langle \partial f_\sigma, \partial f_\sigma \rangle$ vanishes along $f^{-1}(\mathbb{S}^2_\infty)$.

Harmonic maps to the conformal 3-sphere

Conformality at infinity

Let $f : \Sigma \to \mathbb{S}^3$ be a smooth immersion transversal to $\mathbb{S}^2_\infty$.

Definition

f harmonic $:\Leftrightarrow f|_{\Sigma \setminus f^{-1}(\mathbb{S}^2_\infty)} : \Sigma \setminus f^{-1}(\mathbb{S}^2_\infty) \to \mathbb{H}^3_+ \cup \mathbb{H}^3_-$ is harmonic w.r.t. g_{hyp}.

Remark

If f is conformal at infinity and harmonic, then q_{hyp} extends to a holomorphic section in $H^0(\Sigma, K_\Sigma^2)$.

Definition

f orthogonal at infinity $:\Leftrightarrow \forall\, p \in f^{-1}(\mathbb{S}^2_\infty)\ \exists\, v \in T_p\Sigma$ s.t. $Df(p)v \perp T_{f(p)}\mathbb{S}^2_\infty$.

Harmonic maps to the conformal 3-sphere

Example

Willmore cylinder ($q_{hyp} = 0$)

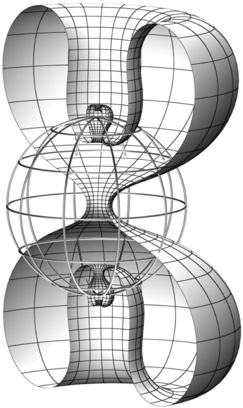

Image by Nick Schmitt (taken from Bobenko et al)

Harmonic maps to the conformal 3-sphere

Equivariance

All notions from above apply to ρ-equivariant harmonic maps

$$f : \tilde{\Sigma} \to \mathbb{S}^3 = \mathbb{H}^3_+ \cup \mathbb{S}^2_\infty \cup \mathbb{H}^3_-$$

for a representation

$$\rho : \pi_1\Sigma \to \mathrm{SL}(2, \mathbb{C})$$

where $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{S}^3$ via

$$\mathrm{SL}(2, \mathbb{C}) \to \mathrm{Isom}_+(\mathbb{H}^3) \to \mathrm{Conf}_+(\mathbb{S}^3)$$

Example

The developing map

$$\mathrm{dev} : \tilde{\Sigma} \to \mathbb{C}\mathrm{P}^1 = \mathbb{H}^2_+ \cup \mathbb{S}^1_\infty \cup \mathbb{H}^2_-$$

of a $\mathbb{C}\mathrm{P}^1$-structure obtained by grafting is harmonic, conformal and orthogonal at infinity (equivariant w.r.t. $\mathrm{hol} : \pi_1\Sigma \to \mathrm{SL}(2, \mathbb{R})$).

Singular solutions to Hitchin's equation

Setting

Let Σ be a compact Riemann surface and $(E, \bar{\partial}_E, \Phi)$ an $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle over Σ, $q = -\det\Phi \in H^0(\Sigma, K_\Sigma^2)$.

Let $\rho : \pi_1\Sigma \to \mathrm{SL}(2,\mathbb{C})$ the corresponding representation.

Assume $\gamma \cong S^1 \subset \Sigma$ is a vertical trajectory of q, i.e. $\exists$ cylinder $C = (-1,1) \times S^1 \ni z$ s.t. $q = dz^2$ (Strebel cylinder).

Question

Is there a singular (along γ) solution of Hitchin's equation such that the associated equivariant harmonic map $f : \tilde{\Sigma} \to \mathbb{S}^3$ is conformal and orthogonal at infinity?

Recall: (A, ϕ) solution to Hitchin's equation w.r.t. background hermitian metric $h \Rightarrow$

$$\nabla = d_A + \phi + \phi^*$$

flat $\mathrm{SL}(2,\mathbb{C})$-connection.

Singular solutions to Hitchin's equation

A model Higgs bundle

Let $C = \mathbb{R} \times S^1$ be the infinite cylinder and $t > 0$. We look at the Higgs bundle

$$\bar{\partial}_E = \bar{\partial}, \quad \Phi_t = \begin{pmatrix} 0 & 1 \\ t^2 & 0 \end{pmatrix} dz$$

on the trivial bundle $E = C \times \mathbb{C}^2$. We wish to solve Hitchin's equation

$$F^{h_t} + [\Phi_t \wedge \Phi_t^{*_{h_t}}] = 0$$

for a hermitian metric h_t which is singular at the circle $\{0\} \times S^1$.

The Ansatz

$$h_t = \begin{pmatrix} e^{u_t} & 0 \\ 0 & e^{-u_t} \end{pmatrix}$$

for a function $u_t : C \to \mathbb{R}$ leads to the scalar PDE

$$\partial_{\bar{z}} \partial_z u_t = e^{2u_t} - t^4 e^{-2u_t}.$$

Singular solutions to Hitchin's equation

A model Higgs bundle

A solution to the PDE

$$\partial_{\bar{z}}\partial_z u_t = e^{2u_t} - t^4 e^{-2u_t}$$

is given by

$$u_t(x) = \frac{1}{2}\log f_t(x) \quad \text{where} \quad f_t(x) = t^2\left(\frac{e^{2tx}+1}{e^{2tx}-1}\right)^2.$$

Hence a solution to Hitchin's equation is given by the metric

$$h_t = \begin{pmatrix} e^{u_t} & 0 \\ 0 & e^{-u_t} \end{pmatrix} = \begin{pmatrix} t\left|\frac{e^{2tx}+1}{e^{2tx}-1}\right| & 0 \\ 0 & t^{-1}\left|\frac{e^{2tx}-1}{e^{2tx}+1}\right| \end{pmatrix}.$$

Note that as $x \to 0$

$$h_t(x) \sim \begin{pmatrix} |x|^{-1} & 0 \\ 0 & |x| \end{pmatrix}$$

Singular solutions to Hitchin's equation

A model Higgs bundle

In unitary gauge we obtain

$$A_t = t\sinh(2tx)^{-1}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} dy \xrightarrow[t\to\infty]{} 0$$

and

$$\phi_t = h_t^{1/2}\Phi_t h_t^{-1/2} = t\begin{pmatrix} 0 & \left|\frac{e^{2tx}+1}{e^{2tx}-1}\right| \\ \left|\frac{e^{2tx}-1}{e^{2tx}+1}\right| & 0 \end{pmatrix} dz.$$

Note that

$$\begin{pmatrix} 0 & \left|\frac{e^{2tx}+1}{e^{2tx}-1}\right| \\ \left|\frac{e^{2tx}-1}{e^{2tx}+1}\right| & 0 \end{pmatrix} dz \xrightarrow[t\to\infty]{} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz.$$

Hence the family (A_t, ϕ_t) is asymptotic to a limiting configuration as $t \to \infty$.

Singular solutions to Hitchin's equation

The model harmonic map

The model harmonic map $f_t : \tilde{C} = \mathbb{C} \to \mathbb{S}^3$ is given by

$$f_t(x,y) = \frac{\begin{pmatrix} 8t^2y^2 + \cosh(4tx) + 1 & \cosh(4tx) + 8ty(ty+i) - 3 \\ \cosh(4tx) + 8ty(ty-i) - 3 & 8t^2y^2 + \cosh(4tx) + 1 \end{pmatrix}}{4\sinh(2tx)}$$

The map is in fact planar and given into $\hat{\mathbb{C}} = \mathbb{H}^2_+ \cup \mathbb{S}^1_\infty \cup \mathbb{H}^2_-$ as

$$f_t(x,y) = (2ty, \sinh(2tx)).$$

Note that $\nabla^g = g^{-1} \circ \nabla \circ g$ is a smooth connection for

$$g = \begin{pmatrix} -|x|^{1/2} & |x|^{-1/2} \\ |x|^{1/2} & |x|^{-1/2} \end{pmatrix}$$

and hence has a well-defined parallel transport across $\{x = 0\}$.

Singular solutions to Hitchin's equation

Gluing

Assume now in addition: q has simple zeroes

Equip the cylinder $C = (-1,1) \times S^1$ with the hyperbolic metric

$$g = \frac{dx^2 + dy^2}{x^2}$$

and extend arbitrarily to $\Sigma \setminus \gamma$.

If $Z = \{p_1, \ldots, p_N\}$ is the zero set of q, glue a limiting configuration (A_∞, ϕ_∞) on $\Sigma \setminus Z$ to a singular model solution (A_t, ϕ_t) on C and the usual model solutions

$$A_t^i = \left(\tfrac{1}{8} + \tfrac{1}{4} r \partial_r v_t(r)\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right),$$

$$\phi_t^i = \begin{pmatrix} 0 & r^{1/2} e^{v_t(r)} \\ r^{1/2} e^{i\theta} e^{-v_t(r)} & 0 \end{pmatrix} dz$$

on balls $B_1(p_i)$.

Obtain approximate solution $(A_t^{app}, \phi_t^{app})$ for t large.

Singular solutions to Hitchin's equation

Theorem

For t sufficiently large there exists $\gamma_t \in H^{2,2}(i\mathfrak{su}(E))$ with $\|\gamma_t\|_{H^{2,2}} = O(e^{-\delta t})$ such that $(A_t, \phi_t) = (A_t^{app}, \phi_t^{app})^{g_t}$ for $g_t = \exp(\gamma_t)$ is a solution of Hitchin's equation

$$F^{A_t} + [\phi_t^* \wedge \phi_t] = 0, \quad \bar{\partial}_{A_t}\phi_t = 0$$

on $M = \Sigma \setminus \gamma$.

Equivalently

Theorem

For t sufficiently large there exists a harmonic metric h_t on $(\bar{\partial}_E, \Phi)$ which is singular along core circle γ of the Strebel cylinder and exponentially close to h_t^{app}.

Singular solutions to Hitchin's equation

Uniformly degenerate/0-differential operators

The metric g on $M = \Sigma \setminus \gamma$ is conformally compact, i.e. $\pm x$ are boundary defining functions for $\partial \bar{M} = \partial_+ \bar{M} \cup \partial_- \bar{M}$ and

$$g = x^{-2} \bar{g}|_M$$

for a metric $\bar{g}$ on $\bar{M}$.

Geometric operators have the form

$$Pf = \sum_{i+j \leq k} a_{ij}(x,y)(x\partial_x)^i (x\partial_y)^j f$$

with coefficients a_{ij} smooth up to the boundary.

If ρ is a boundary defining function, then

$$I_s(P) = \rho^{-s} \circ P \circ \rho^s|_{\partial \bar{M}}$$

is called indicial family, $s \in \mathbb{C}$ indicial root $\Leftrightarrow$ $I_s(P)$ not invertible

Singular solutions to Hitchin's equation

Uniformly degenerate/0-differential operators

Theorem (Mazzeo, Lee)

Let P be a uniformly degenerate elliptic operator of order 2 with indicial radius $R > 0$. If P is formally selfadjoint and coercive at infinity, then

1. $P : H^{k,2}_\delta(M, \mathcal{E}) \to H^{k-2,2}_\delta(M, \mathcal{E})$ *is Fredholm if and only if* $\delta \in (-R, R)$.
2. $P : C^{k,\alpha}_\delta(M, \mathcal{E}) \to C^{k-2,\alpha}_\delta(M, \mathcal{E})$ *is Fredholm if and only if* $\delta \in (\frac{1}{2} - R, \frac{1}{2} + R)$.

In both cases, the index of P is zero and the kernel equals the L^2-kernel of P.

Note that $x^\gamma \in L^2(dx/x^2)$ if and only if $\gamma > \frac{1}{2}$.

We apply this to the linearization of Hitchin's equation
$L = \Delta_A - i * M_\phi : \Gamma(M, i\mathfrak{su}(E)) \to \Gamma(M, i\mathfrak{su}(E))$.

Singular solutions to Hitchin's equation

Uniformly degenerate/0-differential operators

The indicial family for L is given by

$$I_s(L) = s(1-s)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 8 & 5 \\ 5 & -8 \end{pmatrix}$$

and hence the indicial roots by $(1 \pm \sqrt{33})/2$ for the diagonal and $(1 \pm \sqrt{21})/2$ for the off-diagonal part, in particular $R = \sqrt{21}/2$.

This gives

Proposition

1. $L : H^{k,2}_\delta(M, i\mathfrak{su}(E)) \to H^{k-2,2}_\delta(M, i\mathfrak{su}(E))$ *is invertible for* $\delta \in (-R, R)$, *and*
2. $L : C^{k,\alpha}_\delta(M, i\mathfrak{su}(E))) \to C^{k-2,\alpha}_\delta(M, i\mathfrak{su}(E))$ *is invertible for* $\delta \in (\frac{1}{2} - R, \frac{1}{2} + R)$.

Singular solutions to Hitchin's equation

Regularity

Work in progress: improvement to weights $\delta > 0$

Regularity of correction term on $\bar{M}$ determines regularity of equivariant harmonic map to $\mathbb{S}^3$

Compare:

- Li-Tam: $f : \overline{\mathbb{H}}^2 \to \overline{\mathbb{H}}^3$ C^1 such that $f : \mathbb{H}^2 \to \mathbb{H}^3$ harmonic and $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^2$ smooth with non-vanishing energy density $\Longrightarrow f$ $C^{2,\alpha}$ on $\overline{\mathbb{H}}^2$
- Economakis (PhD thesis): Li-Tam assumptions $\Rightarrow f$ is polyhomogeneous, i.e. f admits expansion

$$f \sim h + \sum_{k=0}^{\infty} \sum_{l=0}^{N_k} x^{\nu_k} (\log x)^l \phi_{kl}$$

for locally defined maps h, ϕ_{kl} smooth up to the boundary

Let E be a complex vector bundle over Σ.

$\lambda \in \mathbb{C}$, $\bar{\partial}$ holomorphic structure on E,
$D : \Gamma(\Sigma, E) \to \Gamma(\Sigma, E \otimes K_\Sigma)$ differential operator of order 1

Definition

$(\lambda, \bar{\partial}, D)$ is called λ-connection $:\Leftrightarrow$

1. $D(fs) = \lambda\, \partial f \otimes s + f\, Ds$ for $f \in C^\infty(\Sigma, C)$, $s \in \Gamma(\Sigma, E)$
2. $D\bar{\partial} + \bar{\partial} D = 0$.

$\lambda \neq 0$: flat connection $\nabla^\lambda := \bar{\partial} + \frac{1}{\lambda} D$.

$\lambda = 0$: Higgs bundle $(\bar{\partial}, \Phi := D)$.

Hodge moduli space

$$\mathcal{M}_{Hod}(\Sigma) = \text{space of } \lambda\text{-connections}/\mathcal{G}^{\mathbb{C}}$$

Deligne-Hitchin moduli space

Twistor space

Deligne gluing

$$\Psi\colon \mathcal{M}_{Hod}(\Sigma)_{|\mathbb{C}^*} \to \mathcal{M}_{Hod}(\bar{\Sigma})_{|\mathbb{C}^*}, \quad [\lambda, \bar{\partial}, D] \mapsto [\lambda^{-1}, \lambda^{-1} D, \lambda^{-1}\bar{\partial}]$$

The Deligne-Hitchin moduli space is given by

$$\mathcal{M}_{DH} := \mathcal{M}_{Hod}(\Sigma) \cup_{\Psi} \mathcal{M}_{Hod}(\bar{\Sigma})$$

with $\mathbb{C}^*$-action $t(\lambda, \bar{\partial}, D) = (t\lambda, \bar{\partial}, tD)$, fibration over $\mathbb{CP}^1$.

(A, ϕ) solution of Hitchin's equation $\Rightarrow$

$$\lambda \mapsto (\lambda, \bar{\partial}_A + \lambda\phi^*, \lambda\partial_A + \phi)$$

twistor line.

Work of Heller suggests: Singular solutions to Hitchin's equation give rise to exotic twistor lines