Moduli of SU_2-representations and spherical surfaces

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Joint work with
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Spherical metrics

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A *spherical surface* is a triple $(S, \boldsymbol{x}, h)$.

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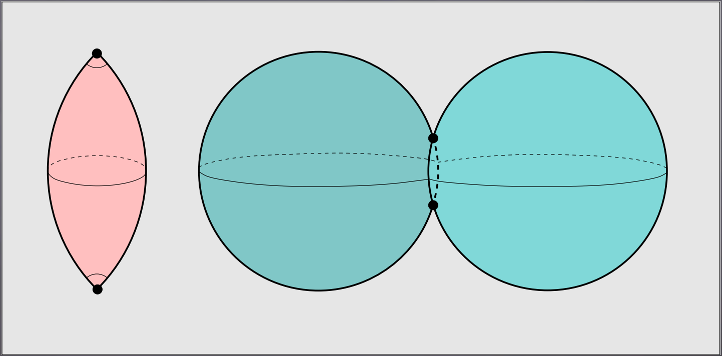

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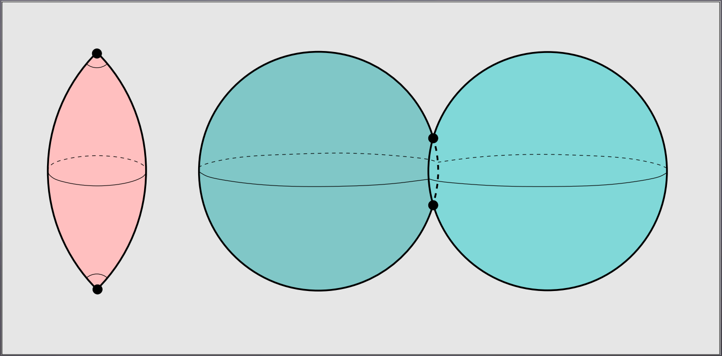

Now on $n \geq 1$ and $(g, n) \neq (0, 1), (0, 2)$

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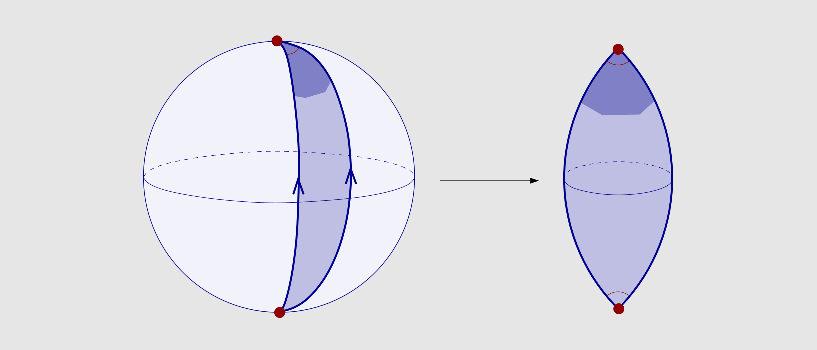

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Note. dev_h is equivariant wrt
monodromy representation $\rho_h : \pi_1(\dot{S}, b) \to \mathrm{SO}_3(\mathbb{R})$

Construction of ρ_h

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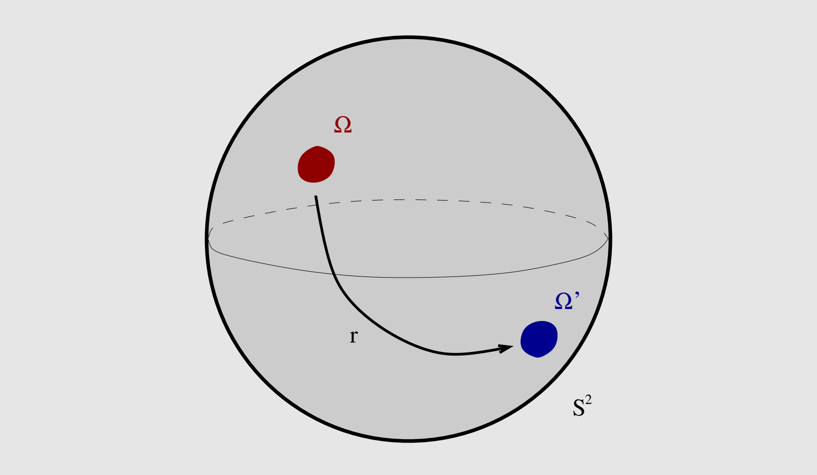
Rigidity property. Given Ω, $\Omega' \subset \mathbb{S}^2$, an isometry $r : \Omega \xrightarrow{\sim} \Omega'$ uniquely extends to $\tilde{r} \in \mathrm{SO}_3(\mathbb{R})$.



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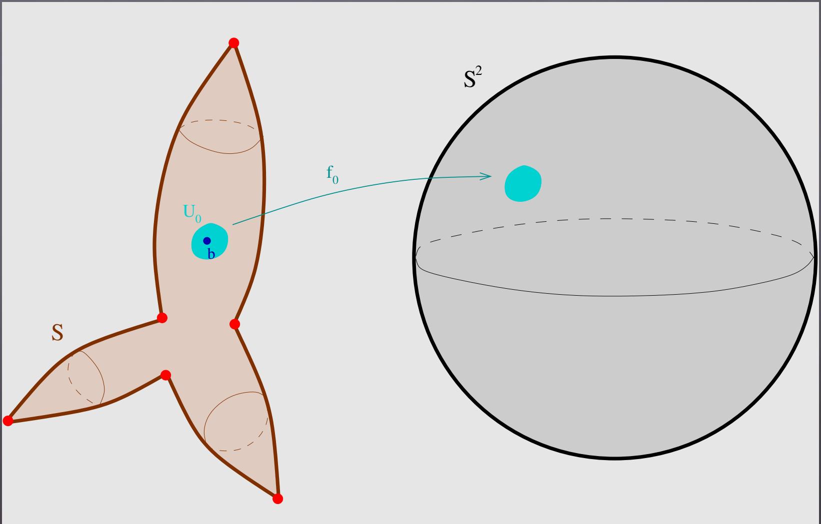
Fix $b \in \dot{S}$, an open contractible neighborhood U_0 and an isometry $f_0 : U_0 \to \mathbb{S}^2$.



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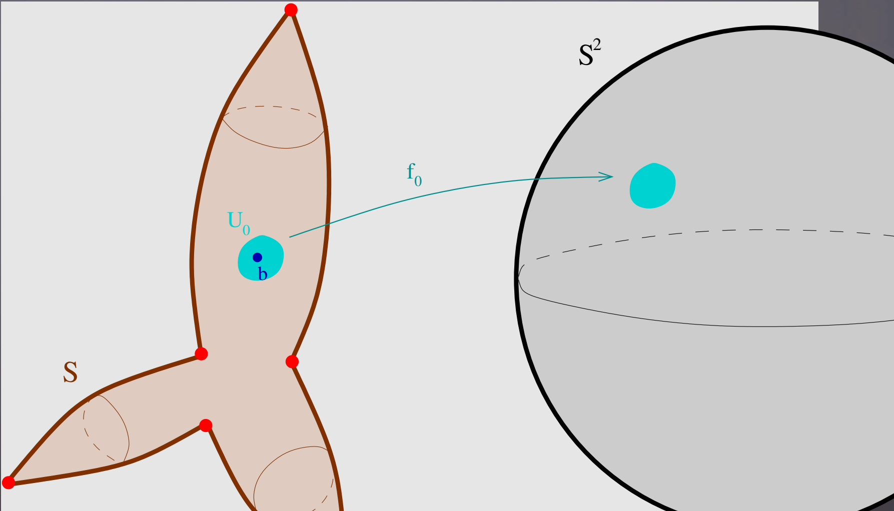

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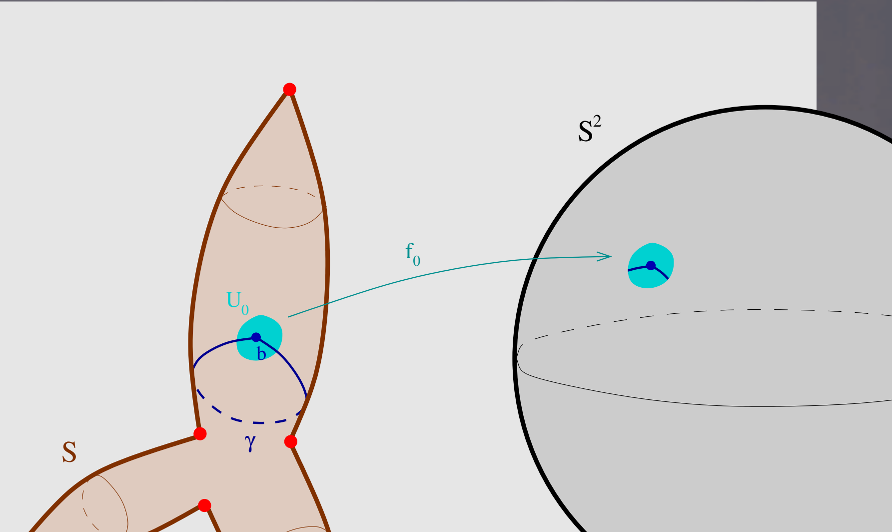
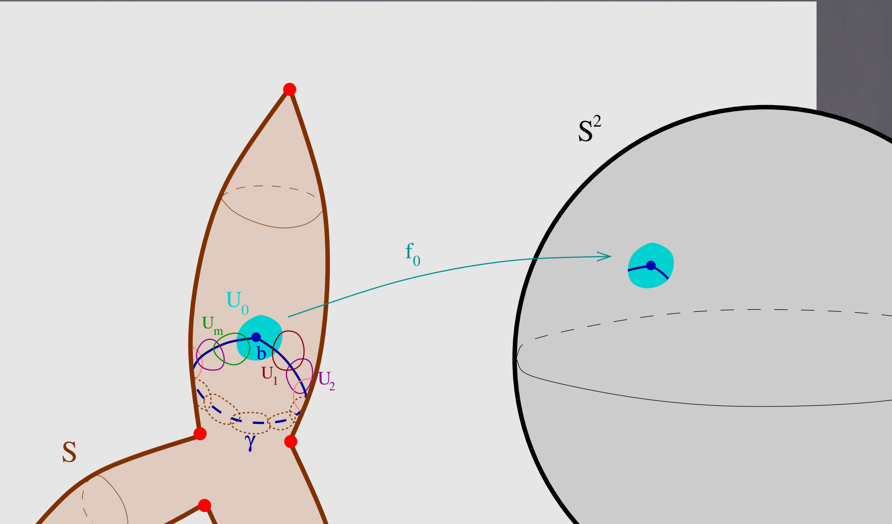
Fix a loop $\gamma \in \pi_1(\dot{S}, b)$.



Construction of ρ_h

Cover γ with $U_0, \ldots, U_m$ contractible open, s.t. $U_0 \cap U_m \neq \emptyset$ and $U_{i-1} \cap U_i \neq \emptyset$.



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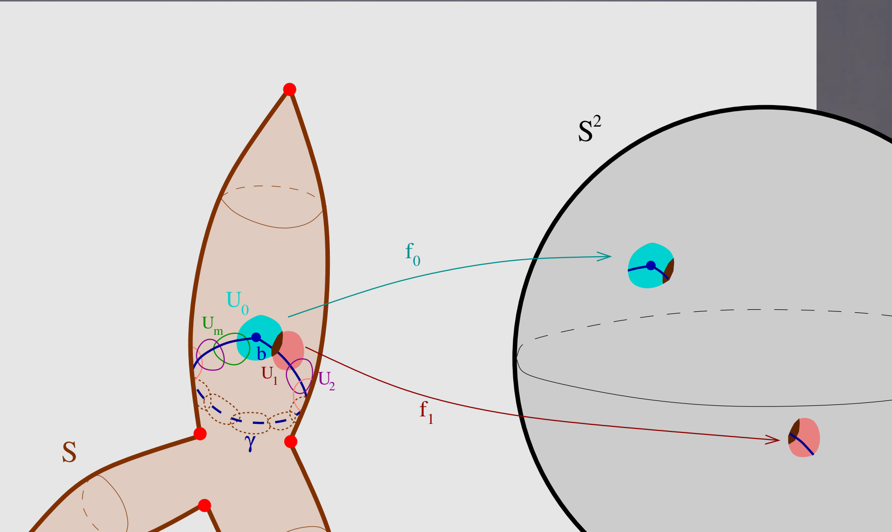
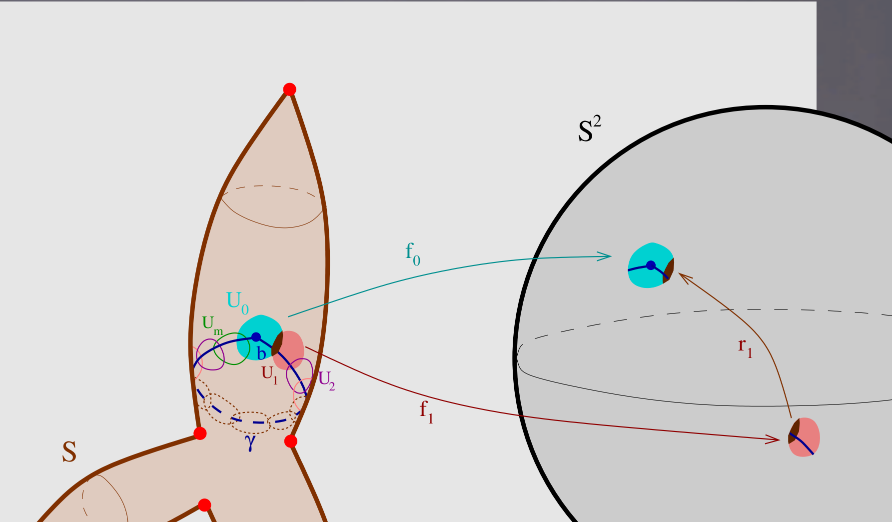
Take an isometry $f_1 : U_1 \to \mathbb{S}^2$.



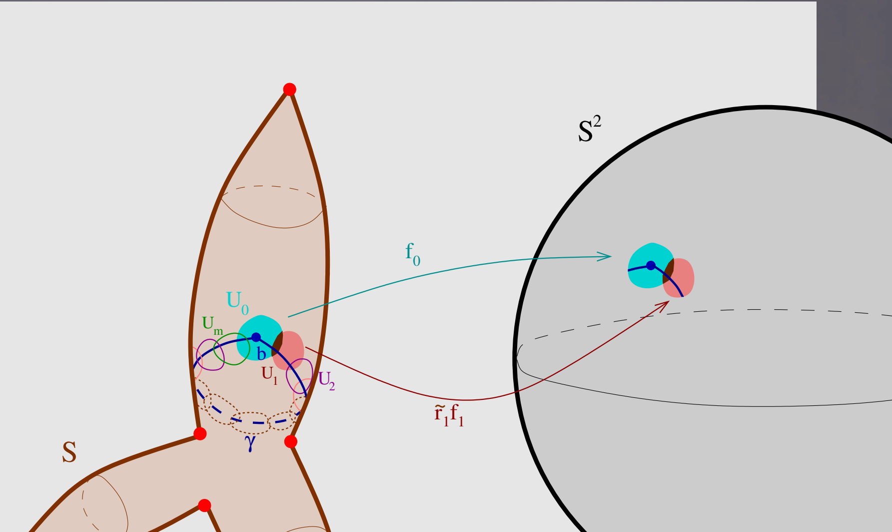
Construction of ρ_h

The isometry $r_1 : f_1(U_1 \cap U_0) \xrightarrow{\sim} f_0(U_1 \cap U_0)$ extends to $\tilde{r}_1 \in \mathrm{SO}_3(\mathbb{R})$.



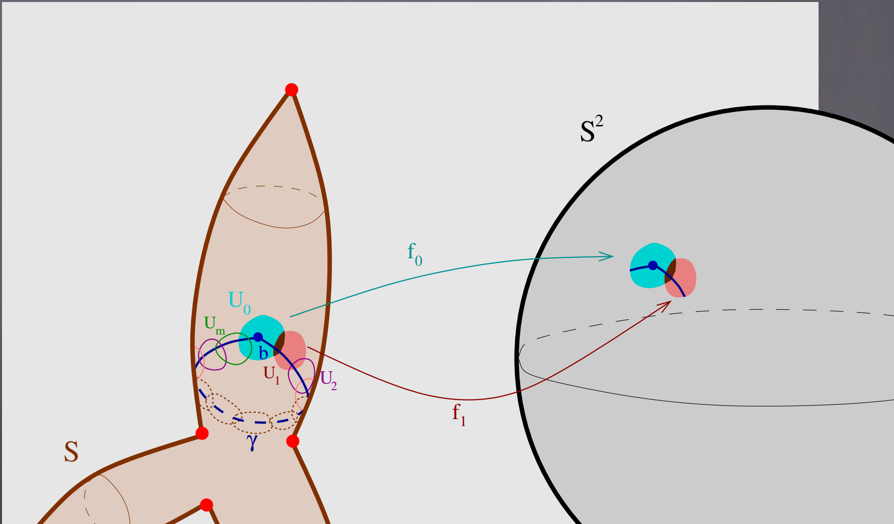
Construction of ρ_h

$f_0 : U_0 \to \mathbb{S}^2$ and $\tilde{r}_1 \circ f_1 : U_1 \to \mathbb{S}^2$ agree on $U_1 \cap U_0$.



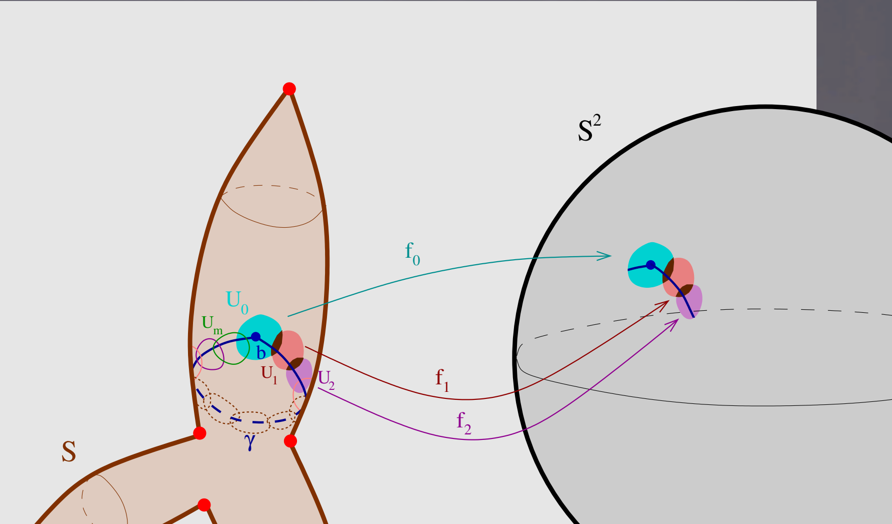
Construction of ρ_h

Replace f_1 by $\tilde{r}_1 \circ f_1$.
So now f_1, f_0 agree on $U_1 \cap U_0$.



Construction of ρ_h

Up to replacing f_2 by $\tilde{r}_2 \circ f_2$, we can assume that f_2, f_1 agree on $U_2 \cap U_1$. And so on...



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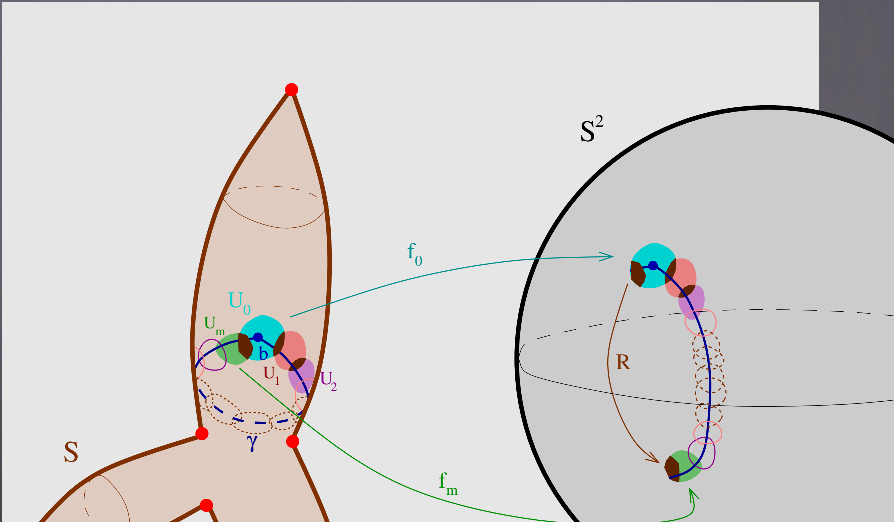
$\exists!\ \tilde{R} \in \mathrm{SO}_3(\mathbb{R})$ such that $\tilde{R} \circ f_0$ agrees with f_m on $U_0 \cap U_m$.



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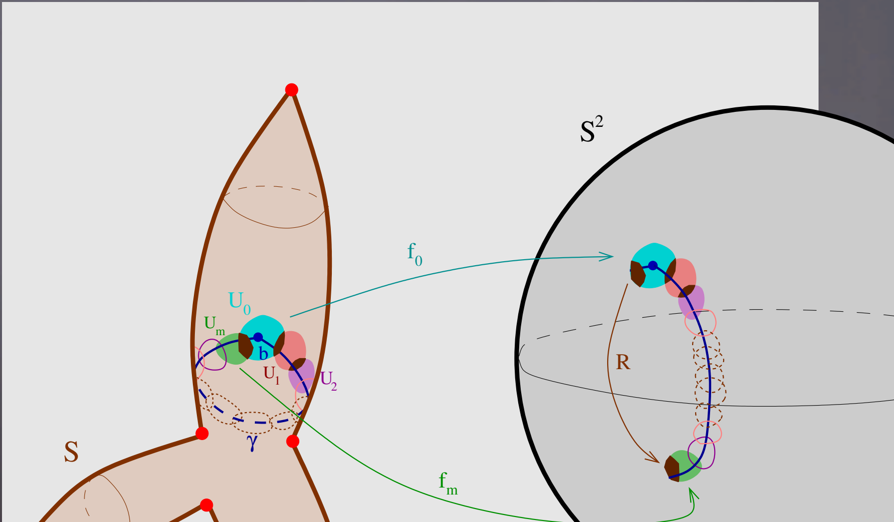
$\exists!\ \tilde{R} \in \mathrm{SO}_3(\mathbb{R})$ such that $\tilde{R} \circ f_0$ agrees with f_m on $U_0 \cap U_m$. Define $\rho_h(\gamma) := \tilde{R}$.



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$\mathcal{MS}_{g,n}(\boldsymbol{\vartheta})$ locus of surfaces with points of angles $2\pi\boldsymbol{\vartheta} = (2\pi\vartheta_1, \ldots, 2\pi\vartheta_n)$

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$F : \mathcal{MS}_{g,n}(\vartheta) \longrightarrow \mathcal{M}_{g,n}$ forgetful map
$F[S, \mathbf{x}, h] := [S, \mathbf{x}, J]$

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locus of $[\rho]$ with $\rho(\beta_j) \in c_j$

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Restriction

$$M_\vartheta : \mathcal{MS}_{g,n}(\vartheta) \xrightarrow{\text{loc}} \mathcal{Rep}_\vartheta(\dot{S}, \mathrm{SU}_2)$$

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Note. If *some* ϑ_j *is integer*, then the monodromy map has positive dimensional fibers

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Def. A decoration for $\rho : \pi_1(\dot{S}, b) \to \mathrm{SU}_2$ is a

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Monodromy is non-elementary

Argument by contradiction:

- Let $\widehat{M}(S, x, h) = (\rho_h, \mathrm{A}_h)$ elementary
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- $V \neq 0$ contradicts $\chi(\dot{S}) < 0$

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Analyticity of representation spaces

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$$\boxed{\mathcal{R}ep(\dot{S}, \mathrm{SU}_2) = \mathcal{H}om(\dot{S}, \mathrm{SU}_2)/\mathrm{SU}_2}$$

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$$R : \; (\mathrm{SU}_2)^{2g+n} \longrightarrow \mathrm{SU}_2$$
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Cor. $\mathcal{MS}_{g,n}(\vartheta)$ homeomorphic to complement of sub CW (orbi)cpx inside finite CW (orbi)cpx

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Special $\mathcal{R}ep_\vartheta$

Special (g, n, ϑ): all $[\rho] \in \mathcal{R}ep_\vartheta(\dot{S}, \mathrm{SU}_2)$ coaxial

Theorem. Special triples are of the following type.

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$^{(\star)}$ $\mathcal{MS}_{g,n}(\boldsymbol{\vartheta}) = \emptyset \iff g = 0$ and $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) < 1$

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$k :=$number of integral entries of $\boldsymbol{\vartheta}$

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Example. For ϑ odd, $\mathcal{MS}_{1,1}(\vartheta)$ disconnected

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Theorem (2019). $\mathrm{NB}_\vartheta(g, n) \neq 0 \implies F$ proper.

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Strategy for non-compactness of $\mathcal{MS}'$

Two different cases:

- **some ϑ_j integral**

  by hand, show that $\inf_{\mathcal{MS}'} \mathrm{sys}_j = 0$
  use local surgery near x_j

- **no ϑ_j integral**

  sys has local minima
  $\implies$ cannot be done by hand

  use Goldman symplectic structure

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