Intermediate complex structure limit

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Plan

- Degeneration of Calabi-Yau metrics
- Sun-Zhang small complex structure limit
- Intermediate complex structure limit
- Optimal transport problem

Reminder

- Calabi-Yau manifold: (compact) Kähler $(X, J, \omega)+$ nowhere vanishing holomorphic volume form Ω, satisfying the PDE

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- These are interesting because they are Ricci flat, *i.e.* satisfies the vaccum Einstein equation $Ric = 0$.
- Notice in particular there is a canonical measure on the manifold.

Major open problem:

Question

What would happen when the complex structure/Kahler class degenerates?

Kahler class degeneration

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Option 1: Fix complex structure, let the Kahler class go to some nef but not ample class on the boundary of the Kahler cone. (Tosatti, ...)
Option 2: Fix the Kahler class, and vary the complex structure. Concretely: Fix some embedding into an ambient Fano manifold (eg. projective embedding), and vary the coefficients of the defining polynomials.

Complex degeneration

Basic dichotomy:

- Noncollapsing case: Under the diameter one normalisation, the volume stays bounded from below.-> Donaldson-Sun theory, metric limit=algebraic limit.
- Collapsing case: Mostly open. Expect: the metric behaviour depends on a non-negative integer $m \leq n$, m=dimension of the essential skeleton.

Complex degeneration

- Sun-Zhang: **Small complex structure limit**.

$$X_t = \{F_0 F_1 + tF = 0\} \subset M = \text{Fano manifold}, \quad \dim M \geq 3.$$

  Here $F_0 = 0, F_1 = 0$ are smooth and transverse ample divisors whose degrees add up to the anticanonical degree of M, and $F = 0$ intersects $\{F_0 = 0\} \cap \{F_1 = 0\}$ transversely.
- **Large complex structure limit** typical example (Fermat family, related to the SYZ conjecture):

$$\{X_0 X_1 \dots X_{n+1} + tF = 0\} \subset \mathbb{P}^{n+1}.$$

Intermediate complex structure limit example:

$$\{F_0 \dots F_m + tF = 0\} \subset M = \text{Fano manifold}.$$

Here $F_i \in H^0(M, L^{d_i})$, with $d_i \geq 1$, and $\sum d_i = d$, $-k_M = dL$. The divisors are suitably generic so that the intersections are transverse.

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- Main interest: How to describe the CY metric at the level of Kähler potentials.

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Sun-Zhang small complex structure limit

- In the m=1 case, Sun-Zhang gave a gluing description of the CY metric.
- The K3 surface case was previously studied by HSVZ.
- The CY metrics are collapsing and the gluing involves a hierarchy of length scales.

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- The asymptotics of the two Tian-Yau spaces are approximated by the **Calabi ansatz**.
- The interpolation between the two Tian-Yau spaces is achieved by the **Ooguri-Vafa type metric**, lying over the open interval.

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- The Calabi ansatz geometrically looks like an iterated fibration: the smallest scale is a circle bundle, over the compact Calabi-Yau manifold $D_0 \cap D_1$, over the 1-dimensional base.
- In the K3 surface special case (HSVZ), the Calabi-Yau $D_0 \cap D_1$ is an elliptic curve, and the Calabi ansatz amounts to a nilmanifold fibration.

Intermediate complex structure limit

Goal: Partially generalise Sun-Zhang to

$$X_t = \{F_0 \dots F_m + tF = 0\} \subset M = \text{Fano manifold.}$$

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- Caveat: this excludes the large complex structure limit case $m = n$.

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- We work with the normalised Kähler class $\frac{1}{|\log|t||}c_1(L)$. This is the natural normalisation for the Kähler potentials to be uniformly L^∞ bounded.
- Caveat: Our result will only describe the CY potential in the C^0-topology limit. To get metric information, it would remain to get C^2-control.

- Comparison with Sun-Zhang: In the $m = 1$ case, our result captures the potential of the Calabi ansatz region at the C^0-level. It is not refined enough to see the details of the Tian-Yau core region or the Ooguri-Vafa type region.

Optimal transport problem rough description:

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- The target is an m-dimensional simplex $\bar{\Delta} \subset \mathbb{R}^{m+1}/\mathbb{R}(1, 1, \ldots 1)$ with a piecewise smooth density function $W(p)$, which can be written explicitly.
- The optimal transport problem produces a convex function u on Δ, such that the gradient ∇u transports the Lebesgue measure on Δ to the target measure $W(p)dp$ on $\bar{\Delta}$.
- The main outcome is that the convex function u satisfies a **Monge-Ampère type equation**

$$\det(D^2 u) W(\nabla u) = \text{const}.$$

- Pick a smooth positive metric h on $L \to M$; the choice does not matter for now. In particular, we can compute the pointwise magnitudes of $F_0, \dots F_m$ etc. Morever, the metric h restricted to $L \to X_t$ allows us to write the CY metrics on $(X_t, \frac{1}{|\log|t||} c_1(L))$ in terms of a potential $\phi_{CY,t}$.

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- There is a canonical way to extend the convex function u to $\mathbb{R}^{m+1}$ (via Legendre transform techniques).
- Ansatz metric:

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- Ansatz metric:

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- Main theorem: $\|\phi_{CY,t} - \phi_t\|_{C^0} \to 0$ as $t \to 0$.

One word on **proof mechanism**: The optimal transport solution allows one to write down an approximately CY metric on X_t (caveat: approximate only on a subset with most of the measure, not everywhere). Then use pluripotential theory arguments to argue this is C^0-close to the actual CY potential.

Slightly more detail: the main ingredients:

- **Algebraic geometry**: find a concrete basis for the space of sections on (X_t, kL). (Tool: Hilbert series)
- **Uniform Fubini-Study approximation** (Tool: Bergman kernel)
- From the **optimal transport** solution, construct some **approximate CY metric** whose volume form is close to being CY away from a subset of small measure (Tool: generalised Calabi ansatz)
- **Kolodziej's technique** of estimates on Kähler potential (Tool: pluripotential theory).

Comment: the generalized Calabi ansatz is a framework which has two well known special cases:

- For m=1, it reduces to the Calabi ansatz
- For m=n, it reduces to semi-flat metrics as in the SYZ conjecture.
- In general, it is some kind of dim reduction for CY under some torus symmetry, and reduces to some real MA type equation, which is why optimal transport appears.

Analytic subtleties

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- We would like to compare the true CY potential with the ansatz potential. Let

$$\phi_{CY,t} - \phi_{ansatz,t} = |\log|t||\psi_t, \quad \min_{X_t} \psi_t = 0.$$

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- This leads to the weak L^1 potential convergence: given any small $\delta > 0$, then for t small enough depending on δ, the CY measure on which

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  is bigger than $1 - \delta$.
- There is a gap between this and the C^0-convergence in our main theorem, and this gap is caused by the one-sidedness.

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Analytic subtleties

- To improve from weak L^1 potential convergence, to C^0_{loc}-convergence on a large open subset (for experts: 'close to the essential skeleton'), the main ingredient is the mean value inequality.
- This can not work globally on X_t, due to the degeneration of the charts in the non-generic region.
- Instead: there is a more delicate argument involving Ohsawa-Takegoshi extension (uniform Fubini-Study approximation), and algebraic-geometry inputs about the space of sections.

Decomposition of sections

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- On the Fano manifold, we can write any section $s \in H^0(M, lL)$ as a finite sum

$$s = \sum F_0^{l_0} \dots F_m^{l_m} s_{l_0,\dots l_m}.$$

Here $l_i \geq 0$ and $d_0 l_0 + \dots d_m l_m \leq l$, and $s_{l_0,\dots l_m} \in H^0(M, (l - \sum d_i l_i)L)$ does not vanish identically on $D_0 \cap \dots \cap D_m$.

Decomposition of sections

- Now when we restrict the section to X_t, there is an extra equation $F_0 \dots F_m = -tF$. This allows one to eliminate $F_0 \dots F_m$ term to leading order. For small t, this means the $F_0^{l_0} \dots F_m^{l_m}$ term satisfies

$$l_i \geq 0, \quad \sum d_i l_i \leq l, \quad \min l_i = 0. \tag{1}$$

- Thus $(-\frac{l_i}{l})_{i=0}^m$ lies in one of the m-simplices $\Delta_0^\vee, \dots \Delta_m^\vee$, eg.

$$\Delta_0^\vee = \{p_0 = 0, p_i \leq 0, \sum_1^m d_i p_i \geq -1\} \subset \mathbb{R}^{m+1}.$$

- Upon the projection to $\mathbb{R}^{m+1}/\mathbb{R}(1, \dots 1)$, the union of these simplices projects homeomorphically to a simplex $\bar{\Delta}^\vee$ with vertices $(0, \dots -\frac{1}{d_i}, \dots 0)$, with a natural triangulation.

Optimal transport problem

- On the simplices $\Delta_j^\vee$ there is a weighting factor

$$W(p) = (1 + \sum_0^m d_i p_i)^{n-m} \geq 0.$$

This can be regarded as a density function on $\bar{\Delta}^\vee$.

- In the optimal transport problem, the domain is the m-simplex with the Lebesgue measure

$$\Delta = \{x_i \geq 0, \sum_0^m x_i \leq 1\} \subset \mathbb{R}^{m+1}.$$

- The target is $\bar{\Delta}^\vee$ with the measure $W(p)dp$. The density is piecewise smooth, due to the issue of the triangulation.

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- In the physics community there is a lot of interest in simulating CY metrics numerically. Traditional methods include Fubini-Study approximations (cf. Donaldson).
- In the intermediate complex structure limit examples, our work suggests a new method: solve the optimal transport problem numerically instead.
- Challenge: Is there a good numerical scheme which detects fine scale features such as Tian-Yau core regions or Ooguri-Vafa type metrics?