Entire curves on holomorphic symplectic varieties

Ljudmila Kamenova

Stony Brook University

Moduli Spaces and Singularities
CRM, Montréal, May 20-24, 2024

Basics

Definition

A hyperkähler manifold, or an irreducible holomorphic symplectic manifold, M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ with σ everywhere non-degenerate.

Remarks.

1. Any holomorphic 2-form σ induces a homomorphism $\tau : \mathcal{T}_X \to \Omega_X$. The form σ is everywhere non-degenerate if and only if τ is bijective.
2. The last condition implies that $\mathcal{K}_X = \mathcal{O}_X$. In particular, $c_1(X) = 0$.
3. From Beauville-Bogomolov's decomposition theorem, irreducible holomorphic symplectic manifolds are building blocks of compact Kähler manifolds with $c_1 = 0$.

Basics

Definition

A hyperkähler manifold, or an irreducible holomorphic symplectic manifold, M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ with σ everywhere non-degenerate.

Remarks.

1. Any holomorphic 2-form σ induces a homomorphism $\tau : \mathcal{T}_X \to \Omega_X$. The form σ is everywhere non-degenerate if and only if τ is bijective.
2. The last condition implies that $\mathcal{K}_X = \mathcal{O}_X$. In particular, $c_1(X) = 0$.
3. From Beauville-Bogomolov's decomposition theorem, irreducible holomorphic symplectic manifolds are building blocks of compact Kähler manifolds with $c_1 = 0$.

Basics

Definition

A hyperkähler manifold, or an irreducible holomorphic symplectic manifold, M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ with σ everywhere non-degenerate.

Remarks.

1. Any holomorphic 2-form σ induces a homomorphism $\tau : \mathcal{T}_X \to \Omega_X$. The form σ is everywhere non-degenerate if and only if τ is bijective.
2. The last condition implies that $\mathcal{K}_X = \mathcal{O}_X$. In particular, $c_1(X) = 0$.
3. From Beauville-Bogomolov's decomposition theorem, irreducible holomorphic symplectic manifolds are building blocks of compact Kähler manifolds with $c_1 = 0$.

Basics

Definition

A hyperkähler manifold, or an irreducible holomorphic symplectic manifold, M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ with σ everywhere non-degenerate.

Remarks.

1. Any holomorphic 2-form σ induces a homomorphism $\tau : \mathcal{T}_X \to \Omega_X$. The form σ is everywhere non-degenerate if and only if τ is bijective.
2. The last condition implies that $\mathcal{K}_X = \mathcal{O}_X$. In particular, $c_1(X) = 0$.
3. From Beauville-Bogomolov's decomposition theorem, irreducible holomorphic symplectic manifolds are building blocks of compact Kähler manifolds with $c_1 = 0$.

K3s

Irreducible holomorphic symplectic manifolds are higher-dimensional analogues of K3 surfaces.

K3

A $K3$ surface is a compact complex surface S with $\mathcal{K}_S = \mathcal{O}_S$ and $b_1(S) = 0$.

Indeed:

1. Kulikov, Siu, Todorov: Every K3 surface is Kähler.
2. Every K3 surface is simply conected (quartic in $\mathbb{CP}^3$).
3. $H^{2,0}(S)$ is generated by a non-degenerate holomorphic form.

K3s

Irreducible holomorphic symplectic manifolds are higher-dimensional analogues of K3 surfaces.

K3

A $K3$ surface is a compact complex surface S with $\mathcal{K}_S = \mathcal{O}_S$ and $b_1(S) = 0$.

Indeed:

1. Kulikov, Siu, Todorov: Every K3 surface is Kähler.
2. Every K3 surface is simply conected (quartic in $\mathbb{CP}^3$).
3. $H^{2,0}(S)$ is generated by a non-degenerate holomorphic form.

K3s

Irreducible holomorphic symplectic manifolds are higher-dimensional analogues of K3 surfaces.

K3

A $K3$ surface is a compact complex surface S with $\mathcal{K}_S = \mathcal{O}_S$ and $b_1(S) = 0$.

Indeed:

1. Kulikov, Siu, Todorov: Every K3 surface is Kähler.
2. Every K3 surface is simply conected (quartic in $\mathbb{CP}^3$).
3. $H^{2,0}(S)$ is generated by a non-degenerate holomorphic form.

K3s

Irreducible holomorphic symplectic manifolds are higher-dimensional analogues of K3 surfaces.

K3

A $K3$ surface is a compact complex surface S with $\mathcal{K}_S = \mathcal{O}_S$ and $b_1(S) = 0$.

Indeed:

1. Kulikov, Siu, Todorov: Every K3 surface is Kähler.
2. Every K3 surface is simply conected (quartic in $\mathbb{CP}^3$).
3. $H^{2,0}(S)$ is generated by a non-degenerate holomorphic form.

Equivallent Definition

Definition

A compact connected $4n$-dimensional Riemannian manifold (M, g) is called *irreducible hyperkähler* if its holonomy group is $Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$.

Remarks.

1. $Sp(n)$ is the subgroup of $GL(n, \mathbb{H})$ that preserves the standard Hermitian form on $\mathbb{H}^n$: $\bar{x}_1^n y_1 + \cdots + \bar{x}_1^n y_1$.
2. If (M, g) is hyperkähler, the quaternions $\mathbb{H}$ act as parallel endomorphisms on the tangent bundle of M. In particular, every $\lambda \in \mathbb{H}$ with $\lambda^2 = -1$ gives rise to an almost complex structure on M. Fix a standard basis $I, J, K = IJ$ of $\mathbb{H}$. Then $\lambda = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$. Therefore, there is a S^2-bundle of complex structures on (M, g), the twistor space.
3. The metric g is Kähler with respect to every $\lambda \in S^2$.

Equivallent Definition

Definition

A compact connected $4n$-dimensional Riemannian manifold (M, g) is called *irreducible hyperkähler* if its holonomy group is $Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$.

Remarks.

1. $Sp(n)$ is the subgroup of $GL(n, \mathbb{H})$ that preserves the standard Hermitian form on $\mathbb{H}^n$: $\bar{x}_1^n y_1 + \cdots + \bar{x}_1^n y_1$.
2. If (M, g) is hyperkähler, the quaternions $\mathbb{H}$ act as parallel endomorphisms on the tangent bundle of M. In particular, every $\lambda \in \mathbb{H}$ with $\lambda^2 = -1$ gives rise to an almost complex structure on M. Fix a standard basis $I, J, K = IJ$ of $\mathbb{H}$. Then $\lambda = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$. Therefore, there is a S^2-bundle of complex structures on (M, g), the twistor space.
3. The metric g is Kähler with respect to every $\lambda \in S^2$.

Equivallent Definition

Definition

A compact connected $4n$-dimensional Riemannian manifold (M, g) is called *irreducible hyperkähler* if its holonomy group is $Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$.

Remarks.

1. $Sp(n)$ is the subgroup of $GL(n, \mathbb{H})$ that preserves the standard Hermitian form on $\mathbb{H}^n$: $\bar{x}_1^n y_1 + \cdots + \bar{x}_1^n y_1$.
2. If (M, g) is hyperkähler, the quaternions $\mathbb{H}$ act as parallel endomorphisms on the tangent bundle of M. In particular, every $\lambda \in \mathbb{H}$ with $\lambda^2 = -1$ gives rise to an almost complex structure on M. Fix a standard basis $I, J, K = IJ$ of $\mathbb{H}$. Then $\lambda = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$. Therefore, there is a S^2-bundle of complex structures on (M, g), the twistor space.
3. The metric g is Kähler with respect to every $\lambda \in S^2$.

Equivallent Definition

Definition

A compact connected $4n$-dimensional Riemannian manifold (M, g) is called *irreducible hyperkähler* if its holonomy group is $Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$.

Remarks.

1. $Sp(n)$ is the subgroup of $GL(n, \mathbb{H})$ that preserves the standard Hermitian form on $\mathbb{H}^n$: $\bar{x}_1^n y_1 + \cdots + \bar{x}_1^n y_1$.
2. If (M, g) is hyperkähler, the quaternions $\mathbb{H}$ act as parallel endomorphisms on the tangent bundle of M. In particular, every $\lambda \in \mathbb{H}$ with $\lambda^2 = -1$ gives rise to an almost complex structure on M. Fix a standard basis $I, J, K = IJ$ of $\mathbb{H}$. Then $\lambda = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$. Therefore, there is a S^2-bundle of complex structures on (M, g), the twistor space.
3. The metric g is Kähler with respect to every $\lambda \in S^2$.

Remarks

1. If (M, g) is hyperkähler, then one can construct a holomorphic non-degenerate 2-form on (M, g, I), namely: $\sigma = \omega_J + i\omega_K$, where $\omega_\lambda = g(\lambda\cdot, \cdot)$ is the corresponding Kähler form. Similary if we fix J or K.
2. If X is holomorphic symplectic, then from Yau's solution of Calabi's problem it follows that there is a unique Ricci-flat metric for any fixed Kähler class. This metric is hyperkähler.
3. An irreducible symplectic holomorphic manifold with a fixed Kähler class is the same object as an irreducible hyperkähler manifold with a fixed metric.

Remarks

1. If (M, g) is hyperkähler, then one can construct a holomorphic non-degenerate 2-form on (M, g, I), namely: $\sigma = \omega_J + i\omega_K$, where $\omega_\lambda = g(\lambda\cdot, \cdot)$ is the corresponding Kähler form. Similary if we fix J or K.
2. If X is holomorphic symplectic, then from Yau's solution of Calabi's problem it follows that there is a unique Ricci-flat metric for any fixed Kähler class. This metric is hyperkähler.
3. An irreducible symplectic holomorphic manifold with a fixed Kähler class is the same object as an irreducible hyperkähler manifold with a fixed metric.

Remarks

1. If (M, g) is hyperkähler, then one can construct a holomorphic non-degenerate 2-form on (M, g, I), namely: $\sigma = \omega_J + i\omega_K$, where $\omega_\lambda = g(\lambda\cdot, \cdot)$ is the corresponding Kähler form. Similary if we fix J or K.
2. If X is holomorphic symplectic, then from Yau's solution of Calabi's problem it follows that there is a unique Ricci-flat metric for any fixed Kähler class. This metric is hyperkähler.
3. An irreducible symplectic holomorphic manifold with a fixed Kähler class is the same object as an irreducible hyperkähler manifold with a fixed metric.

Examples

1. K3 surfaces are the only examples in complex dimension 2.
2. Standard series of examples due to Beauville (1983) (Fujiki, $n = 2$):
   (i) the Hilbert schemes of n points $Hilb^n(S)$, where S is a K3 surface. This is the space of 0-dimensional subspaces Z of length $\dim \mathcal{O}_Z = n$. When S is a smooth connected compact surface, $Hilb^n(S)$ is a smooth connected compact manifold of dimension $2n$ which is the desingularization of $Sym^n(S)$ (Fogarty). $(b_2 = 23)$
   (ii) the generalized Kummer varieties $K^{n+1}(A)$, where A is an abelian surface. This is the fiber over $0 \in A$ of the natural morphism $Hilb^{n+1}A \to S^{n+1}A \to A$. $(b_2 = 7)$
3. O'Grady's two exceptional examples in complex dimensions 10 and 6 with $b_2 = 24$ and $b_2 = 8$ respectively (desingularizations of moduli spaces of semistable sheaves: 1999 and 2000).

Examples

1. K3 surfaces are the only examples in complex dimension 2.
2. Standard series of examples due to Beauville (1983) (Fujiki, $n = 2$):
   (i) the Hilbert schemes of n points $Hilb^n(S)$, where S is a K3 surface. This is the space of 0-dimensional subspaces Z of length $\dim \mathcal{O}_Z = n$. When S is a smooth connected compact surface, $Hilb^n(S)$ is a smooth connected compact manifold of dimension $2n$ which is the desingularization of $Sym^n(S)$ (Fogarty). ($b_2 = 23$)
   (ii) the generalized Kummer varieties $K^{n+1}(A)$, where A is an abelian surface. This is the fiber over $0 \in A$ of the natural morphism $Hilb^{n+1}A \to S^{n+1}A \to A$. ($b_2 = 7$)
3. O'Grady's two exceptional examples in complex dimensions 10 and 6 with $b_2 = 24$ and $b_2 = 8$ respectively (desingularizations of moduli spaces of semistable sheaves: 1999 and 2000).

Examples

1. K3 surfaces are the only examples in complex dimension 2.
2. Standard series of examples due to Beauville (1983) (Fujiki, $n = 2$):
   (i) the Hilbert schemes of n points $Hilb^n(S)$, where S is a K3 surface. This is the space of 0-dimensional subspaces Z of length $\dim \mathcal{O}_Z = n$. When S is a smooth connected compact surface, $Hilb^n(S)$ is a smooth connected compact manifold of dimension $2n$ which is the desingularization of $Sym^n(S)$ (Fogarty). ($b_2 = 23$)
   (ii) the generalized Kummer varieties $K^{n+1}(A)$, where A is an abelian surface. This is the fiber over $0 \in A$ of the natural morphism $Hilb^{n+1}A \to S^{n+1}A \to A$. ($b_2 = 7$)
3. O'Grady's two exceptional examples in complex dimensions 10 and 6 with $b_2 = 24$ and $b_2 = 8$ respectively (desingularizations of moduli spaces of semistable sheaves: 1999 and 2000).

Examples

"New examples"

Any small deformation of an irreducible hyperkähler is also an irreducible hyperkähler.

This is because a small deformation of a compact Kähler manifold is again Kähler (due to Kodaira) and also, $H^{2,0}$ is preserved.

Huybrechts (1997)

Two birational projective irreducible hyperkähler manifolds are deformation equivalent and, hence, diffeomorphic.

Note: One can obtain more examples as deformations and birational transformations of the examples above. However, we are going to consider the examples up to deformation equivalence.

Examples

"New examples"

Any small deformation of an irreducible hyperkähler is also an irreducible hyperkähler.

This is because a small deformation of a compact Kähler manifold is again Kähler (due to Kodaira) and also, $H^{2,0}$ is preserved.

Huybrechts (1997)

Two birational projective irreducible hyperkähler manifolds are deformation equivalent and, hence, diffeomorphic.

Note: One can obtain more examples as deformations and birational transformations of the examples above. However, we are going to consider the examples up to deformation equivalence.

Examples

"New examples"

Any small deformation of an irreducible hyperkähler is also an irreducible hyperkähler.

This is because a small deformation of a compact Kähler manifold is again Kähler (due to Kodaira) and also, $H^{2,0}$ is preserved.

Huybrechts (1997)

Two birational projective irreducible hyperkähler manifolds are deformation equivalent and, hence, diffeomorphic.

Note: One can obtain more examples as deformations and birational transformations of the examples above. However, we are going to consider the examples up to deformation equivalence.

Examples

"New examples"

Any small deformation of an irreducible hyperkähler is also an irreducible hyperkähler.

This is because a small deformation of a compact Kähler manifold is again Kähler (due to Kodaira) and also, $H^{2,0}$ is preserved.

Huybrechts (1997)

Two birational projective irreducible hyperkähler manifolds are deformation equivalent and, hence, diffeomorphic.

Note: One can obtain more examples as deformations and birational transformations of the examples above. However, we are going to consider the examples up to deformation equivalence.

Lagrangian Fibrations

Motivation

In studying the moduli space of K3 surfaces, one tries to understand a class of K3 surfaces that is dense in the moduli space and easier to study. Such classes are: Kummer surfaces, quartics in $\mathbb{P}^3$ and elliptic K3 surfaces. The structure that will generalize to higher dimensions that is most helpful is a fibration by abelian varieties.

Definition

A *Lagrangian* fibration on a 2n-dimensional irreducible hyperkähler manifold is the structure of a fibration over $\mathbb{P}^n$ whose generic smooth fibre is a Lagrangian variety of dimension n.

Lagrangian Fibrations

Motivation

In studying the moduli space of K3 surfaces, one tries to understand a class of K3 surfaces that is dense in the moduli space and easier to study. Such classes are: Kummer surfaces, quartics in $\mathbb{P}^3$ and elliptic K3 surfaces. The structure that will generalize to higher dimensions that is most helpful is a fibration by abelian varieties.

Definition

A *Lagrangian* fibration on a 2n-dimensional irreducible hyperkähler manifold is the structure of a fibration over $\mathbb{P}^n$ whose generic smooth fibre is a Lagrangian variety of dimension n.

Lagrangian Fibrations

Matsushita's theorem

Consider a fibration $f\colon M \to B$ with $0 < \dim B < \dim M$. Then $\dim B = \frac{1}{2}\dim M$ and the general fiber is a Lagrangian abelian variety.

Hwang's theorem

If the base B is smooth, then $B = \mathbb{C}P^n$.

Beauville-Bogomolov-Fujiki form

Given a hyperkähler M, there is a non-degenerate integral form q on $H^2(M, \mathbb{Z})$ of signature $(3, b_2 - 3)$ satisfying Fujiki's relation $\int_M \alpha^{2n} = c \cdot q(\alpha)^n$ for $\alpha \in H^2(M, \mathbb{Z})$, with $c > 0$ a constant depending on the topological type of M. This generalizes the intersection pairing on K3 surfaces.

Lagrangian Fibrations

Matsushita's theorem

Consider a fibration $f\colon M \to B$ with $0 < \dim B < \dim M$. Then $\dim B = \frac{1}{2}\dim M$ and the general fiber is a Lagrangian abelian variety.

Hwang's theorem

If the base B is smooth, then $B = \mathbb{C}P^n$.

Beauville-Bogomolov-Fujiki form

Given a hyperkähler M, there is a non-degenerate integral form q on $H^2(M,\mathbb{Z})$ of signature $(3, b_2 - 3)$ satisfying Fujiki's relation $\int_M \alpha^{2n} = c \cdot q(\alpha)^n$ for $\alpha \in H^2(M,\mathbb{Z})$, with $c > 0$ a constant depending on the topological type of M. This generalizes the intersection pairing on K3 surfaces.

Lagrangian Fibrations

Matsushita's theorem

Consider a fibration $f: M \to B$ with $0 < \dim B < \dim M$. Then $\dim B = \frac{1}{2}\dim M$ and the general fiber is a Lagrangian abelian variety.

Hwang's theorem

If the base B is smooth, then $B = \mathbb{C}P^n$.

Beauville-Bogomolov-Fujiki form

Given a hyperkähler M, there is a non-degenerate integral form q on $H^2(M, \mathbb{Z})$ of signature $(3, b_2 - 3)$ satisfying Fujiki's relation $\int_M \alpha^{2n} = c \cdot q(\alpha)^n$ for $\alpha \in H^2(M, \mathbb{Z})$, with $c > 0$ a constant depending on the topological type of M. This generalizes the intersection pairing on K3 surfaces.

Fibrations and SYZ

Observation

Given $f\colon M \to \mathbb{C}P^n$, and h an ample class on $\mathbb{C}P^n$, then $\alpha = f^*h$ is nef with $q(\alpha) = 0$.

SYZ Conjecture: Bogomolov; Hassett-Tschinkel; etc.

If L is a nef line bundle on M with $q(L) = 0$, then L induces a Lagrangian fibration, as above.

Remark

This conjecture is known for deformations of $\mathrm{K3}^{[n]}$ (Bayer–Macrì; Markman), for deformations of $K_n(A)$ (Yoshioka), for O'Grady's O_6 (Mongardi–Rapagnetta) and O_{10} (Mongardi–Onorati).

Fibrations and SYZ

Observation

Given $f\colon M \to \mathbb{C}P^n$, and h an ample class on $\mathbb{C}P^n$, then $\alpha = f^*h$ is nef with $q(\alpha) = 0$.

SYZ Conjecture: Bogomolov; Hassett-Tschinkel; etc.

If L is a nef line bundle on M with $q(L) = 0$, then L induces a Lagrangian fibration, as above.

Remark

This conjecture is known for deformations of $\mathrm{K3}^{[n]}$ (Bayer–Macrì; Markman), for deformations of $K_n(A)$ (Yoshioka), for O'Grady's O_6 (Mongardi–Rapagnetta) and O_{10} (Mongardi–Onorati).

Fibrations and SYZ

Observation

Given $f\colon M \to \mathbb{C}P^n$, and h an ample class on $\mathbb{C}P^n$, then $\alpha = f^* h$ is nef with $q(\alpha) = 0$.

SYZ Conjecture: Bogomolov; Hassett-Tschinkel; etc.

If L is a nef line bundle on M with $q(L) = 0$, then L induces a Lagrangian fibration, as above.

Remark

This conjecture is known for deformations of $\mathrm{K3}^{[n]}$ (Bayer–Macrì; Markman), for deformations of $K_n(A)$ (Yoshioka), for O'Grady's O_6 (Mongardi–Rapagnetta) and O_{10} (Mongardi–Onorati).

Kobayashi's pseudometric

Definition

The *Kobayashi pseudometric* on M is the maximal pseudodistance d_M such that all holomorphic maps $f : (D, \rho) \to (M, d_M)$ are distance decreasing, where (D, ρ) is the unit disk with the Poincaré metric.

Definition

A manifold is *Kobayashi hyperbolic* if d_M is a metric.

Brody's theorem

Let M be a compact complex manifold. Then M is Kobayashi non-hyperbolic if and only is there exists an entire curve $\mathbb{C} \to M$.

Kobayashi's pseudometric

Definition

The *Kobayashi pseudometric* on M is the maximal pseudodistance d_M such that all holomorphic maps $f\colon (D,\rho) \to (M, d_M)$ are distance decreasing, where (D,ρ) is the unit disk with the Poincaré metric.

Definition

A manifold is *Kobayashi hyperbolic* if d_M is a metric.

Brody's theorem

Let M be a compact complex manifold. Then M is Kobayashi non-hyperbolic if and only is there exists an entire curve $\mathbb{C} \to M$.

Kobayashi's pseudometric

Definition

The *Kobayashi pseudometric* on M is the maximal pseudodistance d_M such that all holomorphic maps $f: (D, \rho) \to (M, d_M)$ are distance decreasing, where (D, ρ) is the unit disk with the Poincaré metric.

Definition

A manifold is *Kobayashi hyperbolic* if d_M is a metric.

Brody's theorem

Let M be a compact complex manifold. Then M is Kobayashi non-hyperbolic if and only is there exists an entire curve $\mathbb{C} \to M$.

Kobayashi's conjectures

Kobayashi's conjectures (1976)

1. For S a K3 surface we have $d_S \equiv 0$.
2. For M hyperkähler we have $d_M \equiv 0$.
3. A hyperkähler manifold M is Kobayashi non-hyperbolic.

Note

Mori-Mukai '82: The first conjecture holds for projective K3 surfaces, using dominating families of (singular) elliptic curves.

K-Verbitsky' 2012

All known hyperkähler manifolds are Kobayashi non-hyperbolic.

Kobayashi's conjectures

Kobayashi's conjectures (1976)

1. For S a K3 surface we have $d_S \equiv 0$.
2. For M hyperkähler we have $d_M \equiv 0$.
3. A hyperkähler manifold M is Kobayashi non-hyperbolic.

Note

Mori-Mukai '82: The first conjecture holds for projective K3 surfaces, using dominating families of (singular) elliptic curves.

K-Verbitsky' 2012

All known hyperkähler manifolds are Kobayashi non-hyperbolic.

Kobayashi's conjectures

Kobayashi's conjectures (1976)

1. For S a K3 surface we have $d_S \equiv 0$.
2. For M hyperkähler we have $d_M \equiv 0$.
3. A hyperkähler manifold M is Kobayashi non-hyperbolic.

Note

Mori-Mukai '82: The first conjecture holds for projective K3 surfaces, using dominating families of (singular) elliptic curves.

K-Verbitsky' 2012

All known hyperkähler manifolds are Kobayashi non-hyperbolic.

Kobayashi's conjectures

Kobayashi's conjectures (1976)

1. For S a K3 surface we have $d_S \equiv 0$.
2. For M hyperkähler we have $d_M \equiv 0$.
3. A hyperkähler manifold M is Kobayashi non-hyperbolic.

Note

Mori-Mukai '82: The first conjecture holds for projective K3 surfaces, using dominating families of (singular) elliptic curves.

K-Verbitsky' 2012

All known hyperkähler manifolds are Kobayashi non-hyperbolic.

Kobayashi's conjectures

Kobayashi's conjectures (1976)

1. For S a K3 surface we have $d_S \equiv 0$.
2. For M hyperkähler we have $d_M \equiv 0$.
3. A hyperkähler manifold M is Kobayashi non-hyperbolic.

Note

Mori-Mukai '82: The first conjecture holds for projective K3 surfaces, using dominating families of (singular) elliptic curves.

K-Verbitsky' 2012

All known hyperkähler manifolds are Kobayashi non-hyperbolic.

Kobayashi's conjectures

Kobayashi's conjectures (1976)

1. For S a K3 surface we have $d_S \equiv 0$.
2. For M hyperkähler we have $d_M \equiv 0$.
3. A hyperkähler manifold M is Kobayashi non-hyperbolic.

Note

Mori-Mukai '82: The first conjecture holds for projective K3 surfaces, using dominating families of (singular) elliptic curves.

K-Verbitsky' 2012

All known hyperkähler manifolds are Kobayashi non-hyperbolic.

Main Theorems

Verbitsky' 2013

M. Verbitsky extended this to all hyperkähler manifolds.

Theorem 1 (K - Lu - Verbitsky' 2014)

Let S be a K3 surface. Then $d_S \equiv 0$.

Theorem 2 (K - Lu - Verbitsky' 2014; K - Lehn' 2022)

Let M be a primitive symplectic variety with $\rho < \max = b_2 - 2$ and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Theorem 3 (K - Lu - Verbitsky' 2014; K - Lehn'2022)

Let M be a primitive symplectic variety with $\rho = b_2 - 2$ and $b_2(M) \geq 7$. Assume the SYZ conjecture holds for all deformations of M. Then $d_M \equiv 0$.

Main Theorems

Verbitsky' 2013

M. Verbitsky extended this to all hyperkähler manifolds.

Theorem 1 (K - Lu - Verbitsky' 2014)

Let S be a K3 surface. Then $d_S \equiv 0$.

Theorem 2 (K - Lu - Verbitsky' 2014; K - Lehn' 2022)

Let M be a primitive symplectic variety with $\rho < \max = b_2 - 2$ and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Theorem 3 (K - Lu - Verbitsky' 2014; K - Lehn'2022)

Let M be a primitive symplectic variety with $\rho = b_2 - 2$ and $b_2(M) \geq 7$. Assume the SYZ conjecture holds for all deformations of M. Then $d_M \equiv 0$.

Main Theorems

Verbitsky' 2013

M. Verbitsky extended this to all hyperkähler manifolds.

Theorem 1 (K - Lu - Verbitsky' 2014)

Let S be a K3 surface. Then $d_S \equiv 0$.

Theorem 2 (K - Lu - Verbitsky' 2014; K - Lehn' 2022)

Let M be a primitive symplectic variety with $\rho < \max = b_2 - 2$ and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Theorem 3 (K - Lu - Verbitsky' 2014; K - Lehn'2022)

Let M be a primitive symplectic variety with $\rho = b_2 - 2$ and $b_2(M) \geq 7$. Assume the SYZ conjecture holds for all deformations of M. Then $d_M \equiv 0$.

Main Theorems

Verbitsky' 2013

M. Verbitsky extended this to all hyperkähler manifolds.

Theorem 1 (K - Lu - Verbitsky' 2014)

Let S be a K3 surface. Then $d_S \equiv 0$.

Theorem 2 (K - Lu - Verbitsky' 2014; K - Lehn' 2022)

Let M be a primitive symplectic variety with $\rho < \max = b_2 - 2$ and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Theorem 3 (K - Lu - Verbitsky' 2014; K - Lehn'2022)

Let M be a primitive symplectic variety with $\rho = b_2 - 2$ and $b_2(M) \geq 7$. Assume the SYZ conjecture holds for all deformations of M. Then $d_M \equiv 0$.

Algebraic non-hyperbolicity (Demailly)

Proposition (Demailly)

Let M be a compact complex Kobayashi hyperbolic manifold. Let h be a Hermitian metric with corresponding $(1,1)$-form ω_h. Then there exists a constant $A > 0$ such that for any non-constant holomorphic map $\phi : C \to M$ from a smooth projective curve of genus g, we have $2g - 2 \geq A \cdot \int_C \phi^* \omega_h$.

Definition (Demailly)

The property above is called algebraic hyperbolicity.

Question

Since algebraic non-hyperbolicity implies Kobayashi non-hyperbolicity, a natural question to ask is: are hyperkähler varieties algebraically non-hyperbolic?

Algebraic non-hyperbolicity (Demailly)

Proposition (Demailly)

Let M be a compact complex Kobayashi hyperbolic manifold. Let h be a Hermitian metric with corresponding $(1,1)$-form ω_h. Then there exists a constant $A > 0$ such that for any non-constant holomorphic map $\phi : C \to M$ from a smooth projective curve of genus g, we have $2g - 2 \geq A \cdot \int_C \phi^* \omega_h$.

Definition (Demailly)

The property above is called algebraic hyperbolicity.

Question

Since algebraic non-hyperbolicity implies Kobayashi non-hyperbolicity, a natural question to ask is: are hyperkähler varieties algebraically non-hyperbolic?

Algebraic non-hyperbolicity (Demailly)

Proposition (Demailly)

Let M be a compact complex Kobayashi hyperbolic manifold. Let h be a Hermitian metric with corresponding $(1,1)$-form ω_h. Then there exists a constant $A > 0$ such that for any non-constant holomorphic map $\phi : C \to M$ from a smooth projective curve of genus g, we have $2g - 2 \geq A \cdot \int_C \phi^* \omega_h$.

Definition (Demailly)

The property above is called algebraic hyperbolicity.

Question

Since algebraic non-hyperbolicity implies Kobayashi non-hyperbolicity, a natural question to ask is: are hyperkähler varieties algebraically non-hyperbolic?

Algebraic non-hyperbolicity

Theorem 4 (K - Verbitsky' 2017)

Let M be a hyperkähler manifold with Picard rank ρ. Assume that either $\rho > 2$ or $\rho = 2$ and the SYZ conjecture holds. Then M is algebraically non-hyperbolic.

Key theorem (K - Verbitsky' 2017)

Let M be a projective hyperkähler manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

Generalization (Bogomolov - K - Verbitsky' 2017)

Let M be a projective manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

Algebraic non-hyperbolicity

Theorem 4 (K - Verbitsky' 2017)

Let M be a hyperkähler manifold with Picard rank ρ. Assume that either $\rho > 2$ or $\rho = 2$ and the SYZ conjecture holds. Then M is algebraically non-hyperbolic.

Key theorem (K - Verbitsky' 2017)

Let M be a projective hyperkähler manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

Generalization (Bogomolov - K - Verbitsky' 2017)

Let M be a projective manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

Algebraic non-hyperbolicity

Theorem 4 (K - Verbitsky' 2017)

Let M be a hyperkähler manifold with Picard rank ρ. Assume that either $\rho > 2$ or $\rho = 2$ and the SYZ conjecture holds. Then M is algebraically non-hyperbolic.

Key theorem (K - Verbitsky' 2017)

Let M be a projective hyperkähler manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

Generalization (Bogomolov - K - Verbitsky' 2017)

Let M be a projective manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

Set-up

Teich

Consider the Teichmüller space $\text{Teich} := \text{Comp}/\text{Diff}^0(M)$ which admits an action by the *mapping class group* $\Gamma := \text{Diff}^+(M)/\text{Diff}^0(M)$. The Teichmüller space is finite-dimensional for M Calabi-Yau. An element $I \in \text{Teich}$ is *ergodic* if the orbit $\Gamma \cdot I$ is dense in Teich, where $\Gamma \cdot I = \{I' \in \text{Teich} : (M, I) \sim (M, I')\}$.

Verbitsky's theorem

If M is hyperkähler and $I \in \mathrm{Teich}$. Then I is ergodic if and only if the corresponding positive oriented 2-plane contains no rational vectors.

Set-up

Teich

Consider the Teichmüller space $\text{Teich} := \text{Comp}/\text{Diff}^0(M)$ which admits an action by the *mapping class group* $\Gamma := \text{Diff}^+(M)/\text{Diff}^0(M)$. The Teichmüller space is finite-dimensional for M Calabi-Yau. An element $I \in \text{Teich}$ is *ergodic* if the orbit $\Gamma \cdot I$ is dense in Teich, where $\Gamma \cdot I = \{I' \in \text{Teich} : (M, I) \sim (M, I')\}$.

Verbitsky's theorem

If M is hyperkähler and $I \in \mathrm{Teich}$. Then I is ergodic if and only if the corresponding positive oriented 2-plane contains no rational vectors.

Upper semicontinuity

Motivation for "ergodicity"

Via Global Torelli for hyperkähler manfolds, one identifies $\mathrm{Teich}_b^0 = SO(3, b_2 - 3)/SO(2) \times SO(1, b_2 - 3) = \mathrm{Per}$. The mapping class group Γ acts on Per, and it turns out that this action is *ergodic* using the Haar measure on Per and Calvin Moore's theorem.

Key Proposition

Let (M, J) denote a complex manifold with $d_{(M,J)} \equiv 0$. Let $I \in \mathrm{Teich}$ be deformation equivalent to J. Assume I is ergodic. Then $d_{(M,I)} \equiv 0$.

Proof

Indeed, consider $\mathrm{diam} : \mathrm{Teich} \to \mathbb{R}_{\geq 0}$, the maximal distance between two points. This is upper semi-continuous. Then $0 \leq \mathrm{diam}(I) \leq \mathrm{diam}(J) = 0$.

Upper semicontinuity

Motivation for "ergodicity"

Via Global Torelli for hyperkähler manfolds, one identifies $\mathrm{Teich}_b^0 = SO(3, b_2 - 3)/SO(2) \times SO(1, b_2 - 3) = \mathrm{Per}$. The mapping class group Γ acts on Per, and it turns out that this action is *ergodic* using the Haar measure on Per and Calvin Moore's theorem.

Key Proposition

Let (M, J) denote a complex manifold with $d_{(M,J)} \equiv 0$. Let $I \in \mathrm{Teich}$ be deformation equivalent to J. Assume I is ergodic. Then $d_{(M,I)} \equiv 0$.

Proof

Indeed, consider $\mathrm{diam} : \mathrm{Teich} \to \mathbb{R}_{\geq 0}$, the maximal distance between two points. This is upper semi-continuous. Then $0 \leq \mathrm{diam}(I) \leq \mathrm{diam}(J) = 0$.

Upper semicontinuity

Motivation for "ergodicity"

Via Global Torelli for hyperkähler manfolds, one identifies $\text{Teich}_b^0 = SO(3, b_2 - 3)/SO(2) \times SO(1, b_2 - 3) = \text{Per}$. The mapping class group Γ acts on Per, and it turns out that this action is *ergodic* using the Haar measure on Per and Calvin Moore's theorem.

Key Proposition

Let (M, J) denote a complex manifold with $d_{(M,J)} \equiv 0$. Let $I \in \mathrm{Teich}$ be deformation equivalent to J. Assume I is ergodic. Then $d_{(M,I)} \equiv 0$.

Proof

Indeed, consider $\mathrm{diam} : \text{Teich} \to \mathbb{R}_{\geq 0}$, the maximal distance between two points. This is upper semi-continuous. Then $0 \leq \mathrm{diam}(I) \leq \mathrm{diam}(J) = 0$.

Double fibrations

Theorem 1

Let S be a K3 surface. Then $d_S \equiv 0$.

Proof of Theorem 1

Case I: $\rho = b_2 - 2$
Then S is projective and $d_S \equiv 0$ by Mori-Mukai.
Case II: $\rho(S, I) < b_2 - 2$
Then I is either ergodic or its orbit is in the "intermediate" case and in both cases we can deform (S, I) to a projective (S, J) whence $d_{(S,I)} \equiv 0$.

Key Theorem

Let M be a primitive symplectic variety, admitting two Lagrangian fibrations associated to non-proportional nef parabolic classes. Then $d_M \equiv 0$.

Double fibrations

Theorem 1

Let S be a K3 surface. Then $d_S \equiv 0$.

Proof of Theorem 1

Case I: $\rho = b_2 - 2$
Then S is projective and $d_S \equiv 0$ by Mori-Mukai.
Case II: $\rho(S, I) < b_2 - 2$
Then I is either ergodic or its orbit is in the "intermediate" case and in both cases we can deform (S, I) to a projective (S, J) whence $d_{(S,I)} \equiv 0$.

Key Theorem

Let M be a primitive symplectic variety, admitting two Lagrangian fibrations associated to non-proportional nef parabolic classes. Then $d_M \equiv 0$.

Double fibrations

Theorem 1

Let S be a K3 surface. Then $d_S \equiv 0$.

Proof of Theorem 1

Case I: $\rho = b_2 - 2$

Then S is projective and $d_S \equiv 0$ by Mori-Mukai.

Case II: $\rho(S, I) < b_2 - 2$

Then I is either ergodic or its orbit is in the "intermediate" case and in both cases we can deform (S, I) to a projective (S, J) whence $d_{(S,I)} \equiv 0$.

Key Theorem

Let M be a primitive symplectic variety, admitting two Lagrangian fibrations associated to non-proportional nef parabolic classes. Then $d_M \equiv 0$.

Double fibrations - Proof

Proof

Suppose we have

$$\pi_i : M \to X_i, i = 1, 2$$

with h_i ample on X_i and α_i its pull back to M. Then $q(\alpha_i) = 0$ and the lattice $\langle \alpha_1, \alpha_2 \rangle$ has signature $(1, 1)$. Since $q(\alpha_1, \alpha_2) \neq 0$ we can compute as follows: Let F_i denote a fiber of π_i, i.e., $[F_i] = \alpha_i^n$. Then

$$[F_1][F_2] = \int_M \alpha_1^n \wedge \alpha_2^n = cq\left(\alpha_1, \alpha_2\right)^n \neq 0.$$

Note that pseudodistances of points in a given fiber is zero. We use this to connect arbitrary pairs of points in M using the two fibration structures.

Proof of main theorem 2

Theorem 2

Let M be hyperkähler with $\rho < \max = b_2 - 2$ and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Idea behind Theorem 2

The locus of Teich consisting of Lagrangian fibrations self-intersects. In the intersection one can choose a deformation with two Lagrangian fibrations as in the Theorem about double fibrations, hence $d_M \equiv 0$. In the case when $\rho < b_2 - 2$, the complex structure is either ergodic or in the intermediate orbit by Verbitsky's classification. In both cases since the complex structure is deformation equivalent to one with vanishing Kobayashi pseudometric, we use the upper semicontinuity Proposition in order to complete the proof.

Proof of main theorem 2

Theorem 2

Let M be hyperkähler with $\rho < \max = b_2 - 2$ and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Idea behind Theorem 2

The locus of Teich consisting of Lagrangian fibrations self-intersects. In the intersection one can choose a deformation with two Lagrangian fibrations as in the Theorem about double fibrations, hence $d_M \equiv 0$. In the case when $\rho < b_2 - 2$, the complex structure is either ergodic or in the intermediate orbit by Verbitsky's classification. In both cases since the complex structure is deformation equivalent to one with vanishing Kobayashi pseudometric, we use the upper semicontinuity Proposition in order to complete the proof.

Proof of main theorem 3

Theorem 3

Let M be hyperkähler with $\rho = b_2 - 2$ and $b_2(M) \geq 7$. Assume the SYZ conjecture holds for all deformations of M. Then $d_M \equiv 0$.

Idea behind Theorem 3

If $\rho = b_2 - 2 \geq 5$ then there exists $z \in \mathrm{Pic}(M)$ with $q(z) = 0, z \neq 0$ (by Meyer's theorem for indefinite latices of rank at least 5). The SYZ conjecture says z gives rise to a Lagrangian fibration. Consider $\gamma \in \Gamma_1 := \mathrm{Aut}(\mathrm{Pic}(M))$ with $z' = \gamma(z) \neq z$. This way we get a *second* Lagrangian fibration. Apply the Theorem about double fibrations.

Proof of main theorem 3

Theorem 3

Let M be hyperkähler with $\rho = b_2 - 2$ and $b_2(M) \geq 7$. Assume the SYZ conjecture holds for all deformations of M. Then $d_M \equiv 0$.

Idea behind Theorem 3

If $\rho = b_2 - 2 \geq 5$ then there exists $z \in \mathrm{Pic}(M)$ with $q(z) = 0, z \neq 0$ (by Meyer's theorem for indefinite latices of rank at least 5). The SYZ conjecture says z gives rise to a Lagrangian fibration. Consider $\gamma \in \Gamma_1 := \mathrm{Aut}(\mathrm{Pic}(M))$ with $z' = \gamma(z) \neq z$. This way we get a *second* Lagrangian fibration. Apply the Theorem about double fibrations.

Singular setting

Definition

A *primitive symplectic variety* is a normal compact Kähler variety X with rational singularities, such that $H^1(X, \mathcal{O}_X) = 0$, and for a symplectic form σ, $H^0(X, \Omega_X^{[2]}) = \mathbb{C}\sigma$.

Rational SYZ conjecture

Let X be a primitive symplectic variety. If a nontrivial movable line bundle L on X satisfies $q_X(L) = 0$, then L induces a *rational Lagrangian fibration* $f : X \dashrightarrow B$, i.e., a meromorphic map to a normal Kähler variety B such that f has connected fibers and its general fiber is a Lagrangian subvariety of X.

Singular setting

Definition

A *primitive symplectic variety* is a normal compact Kähler variety X with rational singularities, such that $H^1(X, \mathcal{O}_X) = 0$, and for a symplectic form σ, $H^0(X, \Omega_X^{[2]}) = \mathbb{C}\sigma$.

Rational SYZ conjecture

Let X be a primitive symplectic variety. If a nontrivial movable line bundle L on X satisfies $q_X(L) = 0$, then L induces a *rational Lagrangian fibration* $f : X \dashrightarrow B$, i.e., a meromorphic map to a normal Kähler variety B such that f has connected fibers and its general fiber is a Lagrangian subvariety of X.

Main theorem - singular case

Theorem (K-Lehn'2022)

Let X be a primitive symplectic variety. Suppose that every primitive symplectic variety which is a locally trivial deformation of X satisfies the rational SYZ conjecture. Then the following hold.

1. If $b_2(X) \geq 5$, then X is non-hyperbolic.
2. If $b_2(X) \geq 7$, then the Kobayashi pseudometric d_X vanishes identically.

Key ideas in the singular case

Finding one Lagrangian fibration is a way to produce non-hyperbolic sub-tori, whose Kobayashi pseudodistance is identically zero, and the bound $b_2(X) \geq 5$ implies that there is a non-trivial parabolic class which gives rise to a rational Lagrangian fibration.

Main theorem - singular case

Theorem (K-Lehn'2022)

Let X be a primitive symplectic variety. Suppose that every primitive symplectic variety which is a locally trivial deformation of X satisfies the rational SYZ conjecture. Then the following hold.

1. If $b_2(X) \geq 5$, then X is non-hyperbolic.
2. If $b_2(X) \geq 7$, then the Kobayashi pseudometric d_X vanishes identically.

Key ideas in the singular case

Finding one Lagrangian fibration is a way to produce non-hyperbolic sub-tori, whose Kobayashi pseudodistance is identically zero, and the bound $b_2(X) \geq 5$ implies that there is a non-trivial parabolic class which gives rise to a rational Lagrangian fibration.

Main theorem - singular case

Key ideas in the singular case - continuation

Finding two transversal Lagrangian fibrations implies vanishing of the Kobayashi pseudodistance by the triangular inequality. Of course, the more Lagrangian fibrations one needs to produce, the more "space" one needs in the Néron-Severi group, and thus one needs the bound $b_2(X) \geq 7$ in the second case.

Another key idea

Given a rational Lagrangian fibration, together with C. Lehn we show that there is either a second (rational) Lagrangian fibration which is transversal to the first one, or there exist nontrivial divisorial contractions. In the latter case, there is some birational model for which the contracted Lagrangian fibration is chain connected by its fibers. In both cases this leads to vanishing of the Kobayashi pseudodistance.

Main theorem - singular case

Key ideas in the singular case - continuation

Finding two transversal Lagrangian fibrations implies vanishing of the Kobayashi pseudodistance by the triangular inequality. Of course, the more Lagrangian fibrations one needs to produce, the more "space" one needs in the Néron-Severi group, and thus one needs the bound $b_2(X) \geq 7$ in the second case.

Another key idea

Given a rational Lagrangian fibration, together with C. Lehn we show that there is either a second (rational) Lagrangian fibration which is transversal to the first one, or there exist nontrivial divisorial contractions. In the latter case, there is some birational model for which the contracted Lagrangian fibration is chain connected by its fibers. In both cases this leads to vanishing of the Kobayashi pseudodistance.

Generalizations - dominability results

Definition

An n-dimensional complex manifold M is dominable by $\mathbb{C}^n$ if there is a holomorphic map $F : \mathbb{C}^n \dashrightarrow M$ such that its Jacobian determinant is not identically zero in the domain of F.

Theorem (K-Lu'2024)

Let M be a primitive hyperkähler variety of dimension $2n$. If M admits two distinct Lagrangian fibration structures, then M is meromorphically dominable by $\mathbb{C}^{2n}$.

Theorem (K-Lu'2024)

Let M be a hyperkähler manifold of dimension $2n$. If M admits a Lagrangian fibration $f : M \to \mathbb{C}P^n$ with no multiple fibers in codimension one, then M is holomorphically dominable by $\mathbb{C}^{2n}$.

Generalizations - dominability results

Definition

An n-dimensional complex manifold M is dominable by $\mathbb{C}^n$ if there is a holomorphic map $F : \mathbb{C}^n \dashrightarrow M$ such that its Jacobian determinant is not identically zero in the domain of F.

Theorem (K-Lu'2024)

Let M be a primitive hyperkähler variety of dimension $2n$. If M admits two distinct Lagrangian fibration structures, then M is meromorphically dominable by $\mathbb{C}^{2n}$.

Theorem (K-Lu'2024)

Let M be a hyperkähler manifold of dimension $2n$. If M admits a Lagrangian fibration $f : M \to \mathbb{C}P^n$ with no multiple fibers in codimension one, then M is holomorphically dominable by $\mathbb{C}^{2n}$.

Generalizations - dominability results

Definition

An n-dimensional complex manifold M is dominable by $\mathbb{C}^n$ if there is a holomorphic map $F : \mathbb{C}^n \dashrightarrow M$ such that its Jacobian determinant is not identically zero in the domain of F.

Theorem (K-Lu'2024)

Let M be a primitive hyperkähler variety of dimension $2n$. If M admits two distinct Lagrangian fibration structures, then M is meromorphically dominable by $\mathbb{C}^{2n}$.

Theorem (K-Lu'2024)

Let M be a hyperkähler manifold of dimension $2n$. If M admits a Lagrangian fibration $f : M \to \mathbb{C}P^n$ with no multiple fibers in codimension one, then M is holomorphically dominable by $\mathbb{C}^{2n}$.

Thanks

Special thanks to the organizers!