Too Many Daves

by Dr. Seuss

Did I ever tell you that Mrs. McCave
Had twenty-three sons, and she named them all Dave?

Well, she did. And that wasn't a smart thing to do.
You see, when she wants one, and calls out "Yoo-Hoo!
Come into the house, Dave!" she doesn't get one.
All twenty-three Daves of hers come on the run!

This makes things quite difficult at the McCaves'
As you can imagine, with so many Daves.
And often she wishes that, when they were born,
She had named one of them Bodkin Van Horn.
And one of them Hoos-Foos. And one of them Snimm.
And one of them Hot-Shot. And one Sunny Jim.
Another one Putt-Putt. Another one Moon Face.
Another one Marvin O'Gravel Balloon Face.
And one of them Zanzibar Buck-Buck McFate...

But she didn't do it. And now it's too late.

from *The Sneetches and Other Stories* by Dr. Seuss

Gravitational Instantons:
The Tesserons Landscape

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Moduli Spaces and Singularities
CRM
May 23, 2024

Outline

- Terminology: Gravitational Instantons $\supsetneq$ Tesserons
- Classification
- History of Examples
- Presenting as Moduli Spaces:
  “Modularization”, “Incorporation”, and Monopolization
- Parameter Space of all Tesserons:
  - Horizontal limits
  - Vertical Limits

Terminology

Volume 60A, number 2 PHYSICS LETTERS 7 February 1977

GRAVITATIONAL INSTANTONS

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Received 17 January 1977

By analogy with Yang–Mills' theory, gravitational instantons are defined to be solutions of the classical Einstein equations which are non-singular on some section of complexified spacetime and in which the curvature dies away at large distances. The Schwarzschild and Taub-NUT solutions are simple examples, the latter being self-dual. A many Taub-NUT solution is also given. The significance of the two integrals in the curvature which are pure divergences is discussed.

Examples: Euclidean Schwarzschild solution,
Taub-NUT, multi-Taub-NUT, Eguchi-Hanson
$\mathbb{C}P^2$ (?)

Belavin-Burlankov '76, Eguchi-Freund '76, Hawking '77, Gibbons-Hawking '78

In the early Euclidean quantum gravity papers
a **Gravitational Instanton** is a complete Einstein Riemannian 4-manifold with finite Euler characteristic and signature.

All of the spaces below are required to be complete Riemannian 4-manifolds that are not flat.

- Gravitational Instanton:

$$Ric_{\mu\nu} = \Lambda g_{\mu\nu} \ , \quad \chi \text{ and } \tau \text{ finite.}$$

- Self-dual Gravitational Instanton:

$$Rm = * Rm \ , \ p_1(TM) = \frac{1}{8\pi^2}\int_M \mathrm{tr}\ Rm \wedge Rm < \infty.$$

- Tesseron:

  Non-compact Hyperkähler manifold with finite Pontrjagin number.

Note: Self-duality implies

$$p_1(TM) < \infty \quad \Leftrightarrow \quad \|\mathrm{Rm}\|_{L^2} < \infty.$$

Tesserons $\subsetneq$ Self-Dual Gravitational Instantons $\subsetneq$ Gravitational Instantons

Tesserons are distinguished by their asymptotic **Volume Growth:**

A ball of geodesic ball of radius R

Space	Vol (B_R)
ALE	R^4
ALF	R^3
ALG	R^2
ALH	R, $R^{4/3}$

Classification of Tesserons was recently completed:

- ALE: Kronheimer ‘89
- ALF: Minerbe ’07,’08
- ALG & ALH: G. Chen and X.-X. Chen ‘15; G.Chen and Viaclovsky ‘21
- ALG*: G. Chen and Viaclovsky ’21, Sun, Zhang ’21
- ALH*: Hein, Sun, Viaclovsky, Zhang ’21; Collins, Jacob, Lin ’21; Lee, Lin ‘22

Asymptotic Model

A hyperkahler 4-mld with a triholomorphic isometry has a Gibbons-Hawking form:

$$g = V\vec{x}^2 + \frac{(d\tau + \omega)^2}{V}, \text{ where } *_3 dV = d\omega,$$

All tesserons' model ends have (locally):

- ALE $V = \frac{N}{2|\vec{x}|}$
- ALF $V = \ell + \frac{N}{2|\vec{x}|}$
- ALG $V = C + \frac{N}{2}\ln(x_1^2 + x_2^2)$
- ALH $V = C + Nx_1$

Current literature distinguishes:

ALG* and ALH* are spaces with $N \neq 0$, and

ALG and ALH are with $N = 0$ (locally constant fiber).

Prototypical example: | **Asymptotic metric:**

- ALE $V = \frac{N}{2|\vec{x}|}$

$\mathbb{R}^4$ metric in 'radial coordinates'

$$g = \frac{1}{2x} d\vec{x}^2 + 2x(d\theta + \omega)^2$$

Circle fibration (growing circle): Quotient: $\mathbb{R}^4/\Gamma$, $\Gamma \subset SU(2)$.

- ALF $V = \ell + \frac{N}{2|\vec{x}|}$

The **Taub-NUT**:

$$g = (\ell + \frac{1}{2x}) d\vec{x}^2 + \frac{(d\theta + \omega)^2}{\ell + \frac{1}{2x}}$$

Circle fibration (with stabilizing circle):

- ALG $V = C + \frac{N}{2} \ln(x_1^2 + x_2^2)$

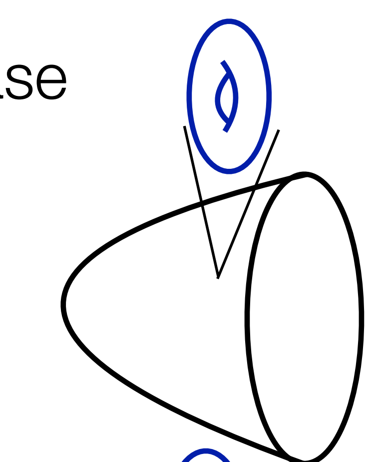
(ALG* if $N \neq 0$)

Elliptic Fibrations:

$$g = \tau_2 dz d\bar{z} + \frac{|d\theta_a + \tau d\theta_b|^2}{\tau_2},$$

$$\tau = \tau_1 + i\tau_2 = C + N\frac{i}{2\pi} \ln z$$

Over cone base

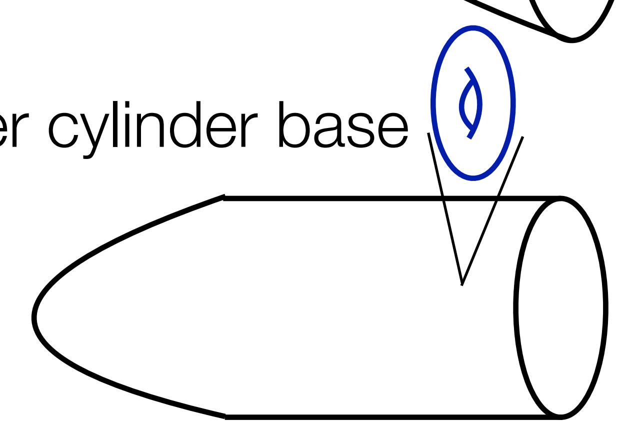


- ALH $V = C + Nx_1$

(ALH* if $N \neq 0$)

$$\tau = \tau_1 + i\tau_2 = C + iNz$$

Over cylinder base



Classification

ALE: $\mathbb{R}^4$, $A_{k\geq 1}$, $D_{k\geq 4}$, and E_6, E_7, E_8

$A_0 = \mathbb{R}^4$,

$A_1 = T^*\mathbb{P}^1 =$ Eguchi-Hanson

ALF: $\mathbb{R}^3 \times S^1$, $A_{k\geq 0}$ and $D_{k\geq 0}$

$A_0 =$ Taub-NUT

$A_k = (k+1)$-centered multi-Taub-NUT

$D_0 =$ Atiyah-Hitchin,

$D_1 =$ deformation of double cover of D_0

$\mathbf{D_2} =$ deformation of $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$

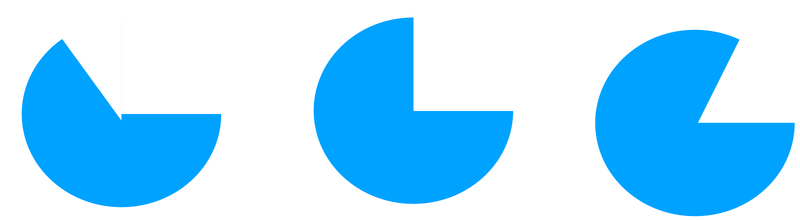
ALG:

& ALG*

$\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$

$\mathbb{R}^2 \times T^2$, D_0, D_1, D_2, D_3, $\mathbf{D}_4$,

Ш₃ , Ш₂ , Ш₁



ALH:

& ALH*

$\mathbb{R} \times T^3$,

E_0, E_1,

$\tilde{E}_1$,

E_2, E_3, E_4, E_5, E_6, E_7, E_8, $\frac{\mathbf{1}}{\mathbf{2}}\mathbf{K3}$

Naive Parameter Count

m_n denotes n real parameters specifying the form of infinity and m "interior" parameters.

ALE: $\mathbb{R}^4$, $A_{k\geq 1}$, $D_{k\geq 4}$, and $E_{k=6,7,8}$.

$(0)_0$ $(3k)_0$ $(3k)_0$ $(3k)_0$

Note: additional isometries reduce this Naive count, e.g. A_1 ALE — $(1)_0$ and A_1 ALF — $(1)_1$

ALF: $\mathbb{R}^3 \times S^1$, $A_{k\geq 0}$ and $D_{k\geq 0}$

$(0)_1$ $(3k)_1$ $(3k)_1$

ALG:

& ALG*

$\mathbb{R}^2 \times T^2$, $(0)_3$

I^*_{4-k} $D_{k=0,1,2,3}$ $(3k)_3$

I^*_0 $\mathbf{D}_4$, $(3k)_3$

$\mathbf{E}_{k=6,7,8}$ $(3k)_1$

Ш$_{k=1,2,3}$ $(3k-1)_1$ (?)

i.e. IV^* $\mathbf{E}_6$, III^* $\mathbf{E}_7$, II^* $\mathbf{E}_8$

IV Ш$_3$, III Ш$_2$, II Ш$_1$

ALH:

& ALH*

$\mathbb{R} \times T^3$, $(0)_7$

E_0, E_1, $\tilde{E}_1$,

E_2, E_3, E_4, E_5, E_6, E_7, E_8, $(3k)_3$

$\frac{1}{2}\mathbf{K3}$ $(3\text{x}8)_7$

More terminology

In relating these spaces, studying their metric, and exploring gauge theory on them it is extremely useful to realize each as a gauge theory moduli space.

For example, all of D-type ALF metrics were explicitly found using their realization as moduli spaces of monopoles.

Realizing a tesseron as a gauge theory moduli space endows it with a lot of interesting structure: families of tautological bundles, connections associated to them, various operators, etc.

An abstract point in a tesseron acquires a body (gauge fields) in some (4d, 3d, etc) world,

so, let us call it "incorporation".

History of Incorporation

- ALE space = moduli spaces of a quiver = $\mathscr{M}$(quiver). Kronheimer '89

- ALF spaces:
  D_0 ALF = Atiyah-Hitchin space = $\mathscr{M}$(rk 2 Nahm equations). Atiyah-Hitchin '78, Nahm '80
  A_k ALF = $\mathscr{M}$(rk (1,k) Nahm Equations).
  D_k ALF = $\mathscr{M}$(rk (2,k) Nahm Equations). Ch-Kapustin '98

- ALG spaces:
  D_k ALG = $\mathscr{M}$(rk 2 Hitchin Equations on $\mathbb{C}P^1$ with 2, 3, or 4 poles). Ch-Kapustin '00
  All ALG = $\mathscr{M}$(Hitchin systems on $\mathbb{C}P^1$ with 3 or 4 singularities) Boalch '12 ("Modularization")

- ALH spaces:
  $E_0,\ldots,E_6$ ALH = $\mathscr{M}$(Monopoles on $\mathbb{R} \times T^2$ w/ simple sings). Ch-Ward '12, Ch, Ch-Cross '19
  E_7 & E_8 ALH = $\mathscr{M}$(Sing. Monopoles on $\mathbb{R} \times T^2$ w/ sings). Thomas Harris '24

Moral:
ALE - Quivers
ALF - Nahm or Bows
ALG - Hitchin or Slings
ALH - Monowalls

This makes some horizontal connections between these moduli spaces apparent, but leaves others obscure.
Vertical connections remain obscure, though interesting.

More terminology:

Let us take a more uniform view of ALL tesserons, by realizing all of them as moduli spaces of monopoles in our 3-dimensional space.

Call it monopolization.

Monopolization

The very first step along this path was taken by Atiyah and Hitchin:

D_0 ALF = Atiyah-Hitchin space = $\mathscr{M}$(centered 2 SU(2) monopoles)

$\mathbb{R}^3 \times S^1$ = $\mathscr{M}$(1 SU(2) monopole)

A_k ALF = $\mathscr{M}$(1 U(2) monopole with k+1 Dirac singularities)

D_k ALF = $\mathscr{M}$(centered 2 U(2) monopoles with k simple Dirac singularities)

First: Let us relate all these ALF spaces (via Gromov-Hausdorff convergence) and relate them to ALE spaces.

Singular Monopoles

Simple Dirac singularities at marked points $p_1^-, \ldots, p_{v_-}^-$ and $p_1^+, \ldots, p_{v_+}^+$: $\Phi = i\begin{pmatrix} \pm\frac{1}{2|\vec{x}-\vec{p}_\sigma^\pm|} & 0 \\ 0 & 0_{n-1,n-1}\end{pmatrix} + O(1)$

Note: More generally the charge is any cocharacter of the gauge group.

Boundary conditions:

$\mathbb{R}^3$

$$\Phi^g(\vec{x}) = \frac{i}{2\pi}\begin{pmatrix} \lambda_1 - \frac{m_1}{2|\vec{x}|} & 0 \\ 0 & -\lambda_2 + \frac{m_2}{2|\vec{x}|}\end{pmatrix} + O(r^{-2})$$

Finite energy => center of mass is a “modulus”

$\mathbb{R}^2 \times S^1$

Coordinates $z = x + iy,\ \varphi$

$$\Phi^g = \frac{i}{2\pi}\mathrm{diag}(v_j + q_j \log|z| + \mathrm{Re}\frac{\mu_j}{z}) + O(1/|z|^2)$$

Infinite energy => center of mass is fixed

$$A^g = \frac{1}{2\pi}\mathrm{diag}((q_j \arg z + [b_j + \mathrm{Im}\frac{\mu_j}{z}])d\theta + \alpha_j d\arg z) + O(|z|^{-2})$$

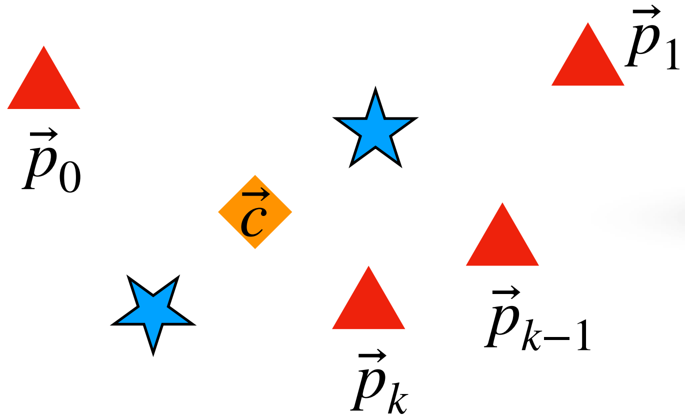
$\mathbb{R} \times T^2$

Coordinates x, θ, φ

$$\Phi^g = \frac{i}{2\pi}\mathrm{diag}(Q_j^\pm x + M_j^\pm) + O(1/x)$$

$$A^g = -\frac{i}{2\pi}\mathrm{diag}(Q_j^\pm \theta d\varphi + \chi_{j,\theta}^\pm d\theta + \chi_{j,\varphi}^\pm d\varphi) + O(1/x)$$

Infinite energy => center of mass is fixed



Monopoles in $\mathbb{R}^3$

Moduli Space:

- A single monopole moduli: position in $\mathbb{R}^3$ and phase in S^1

$(\sqrt{\lambda}\mathbb{R}^3) \times S^1_{\frac{1}{\sqrt{\lambda}}}$

$$\Phi^g(\vec{x}) = \frac{i}{2\pi}\begin{pmatrix} \lambda - \frac{1}{2|\vec{x}|} & 0 \\ 0 & -\lambda + \frac{1}{2|\vec{x}|} \end{pmatrix} + O(r^{-2})$$

- A single monopole with $k+1$ simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_k \in \mathbb{R}^3$

A_k ALF = multi-Taub-NUT with NUTs at $\vec{p}_0, \ldots, \vec{p}_k$

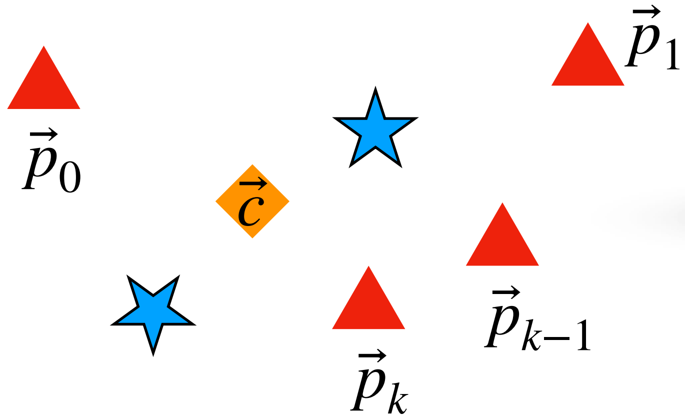
- Two monopoles in $\mathbb{R}^3$: two positions and two phases => 8 dim moduli space with triholomorphic isometry

Two centered monopoles in $\mathbb{R}^3$ with center at $\vec{c} \in \mathbb{R}^3$ — D_0 ALF = Atiyah-Hitchin

Two centered monopoles with k simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_{k-1} \in \mathbb{R}^3$ — D_k ALF

This picture leads to direct relations between these spaces!

Relations between ALF Spaces

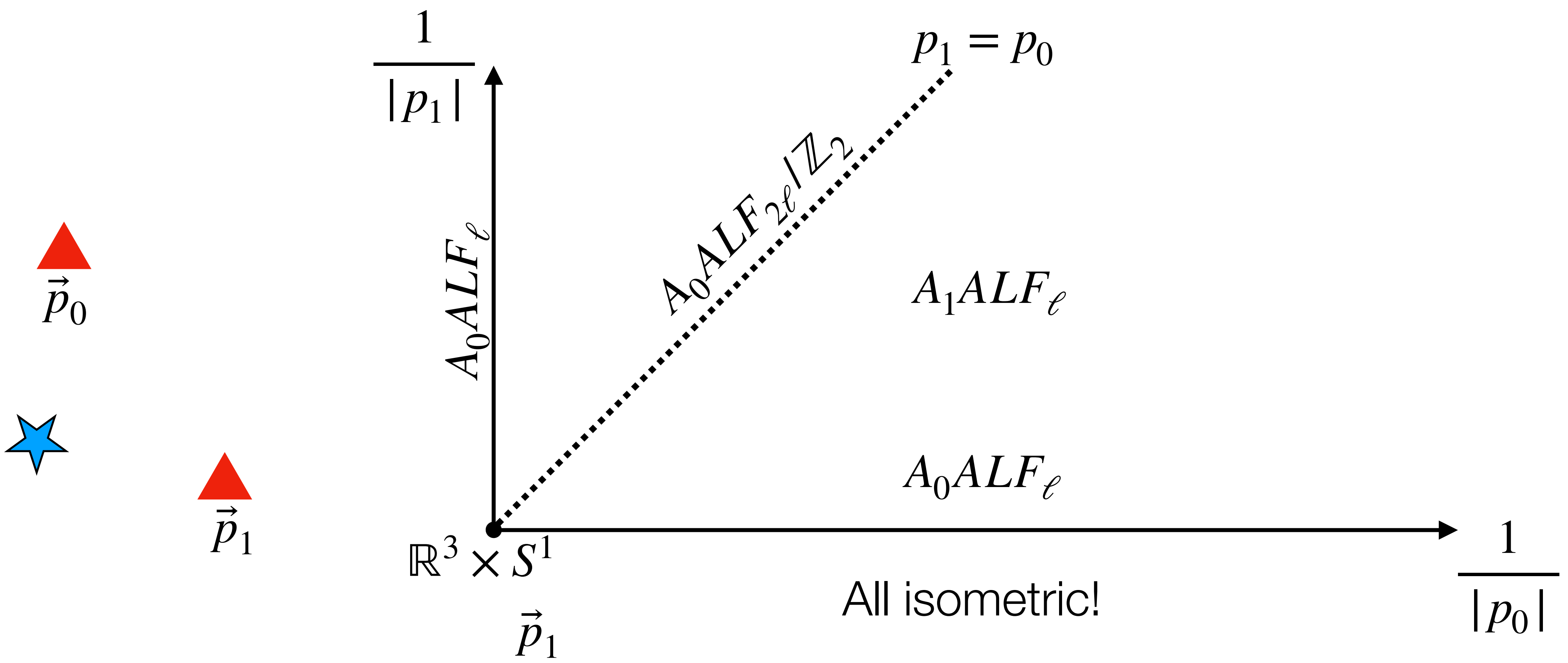


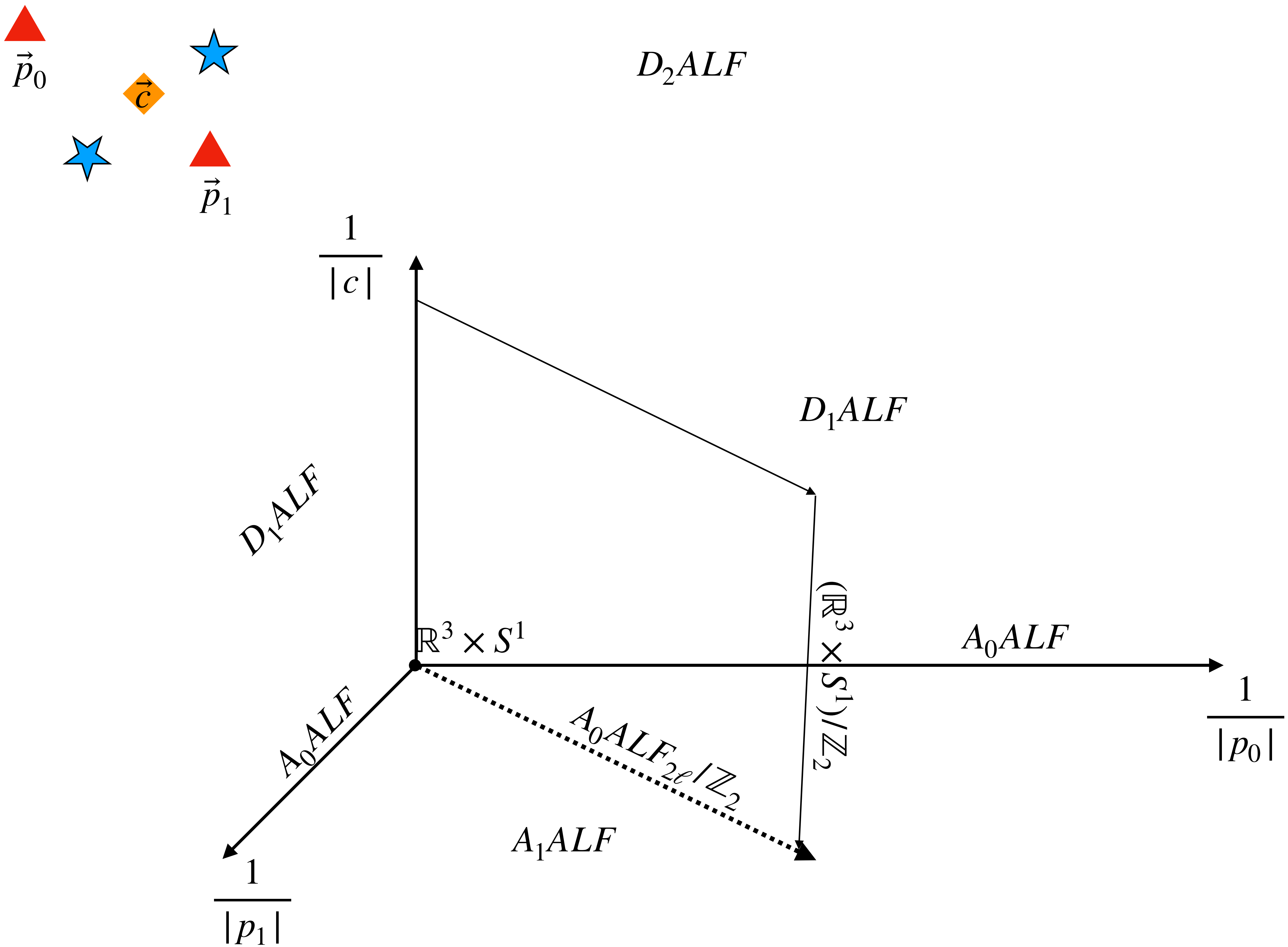
A single with $k+1$ simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_k \in \mathbb{R}^3$ — A_k ALF

Two centered monopoles with k simple Dirac singularities $\vec{p}_0, \ldots, \vec{p}_{k-1} \in \mathbb{R}^3$ — D_k ALF

$$\begin{array}{ccccccccccc}
\to A_k & \xrightarrow{p_k\to\infty} & A_{k-1} & \xrightarrow{p_{k-1}\to\infty} & \ldots \to A_1 & \xrightarrow{p_1\to\infty} & A_0 & \xrightarrow{p_0\to\infty} & \mathbb{R}^3\times S^1 \\
\uparrow{\scriptstyle c\,\to\,\infty} & & \uparrow{\scriptstyle c\,\to\,\infty} & & \uparrow{\scriptstyle c\,\to\,\infty} & & \uparrow{\scriptstyle c\,\to\,\infty} & & \uparrow{\scriptstyle c\,\to\,\infty} \\
\to D_{k+1} & \xrightarrow{p_k\to\infty} & D_k & \xrightarrow{p_{k-1}\to\infty} & \ldots \to D_2 & \xrightarrow{p_1\to\infty} & D_1 & \xrightarrow{p_0\to\infty} & D_0
\end{array}$$

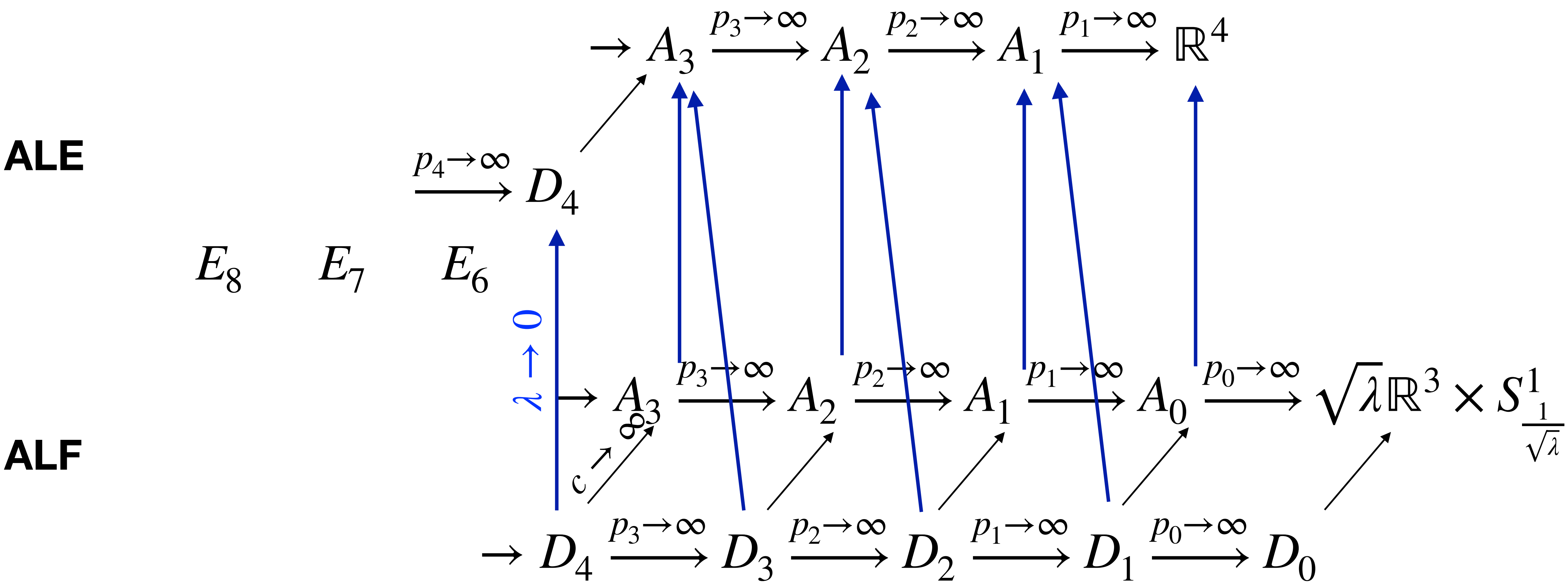
Very schematically:



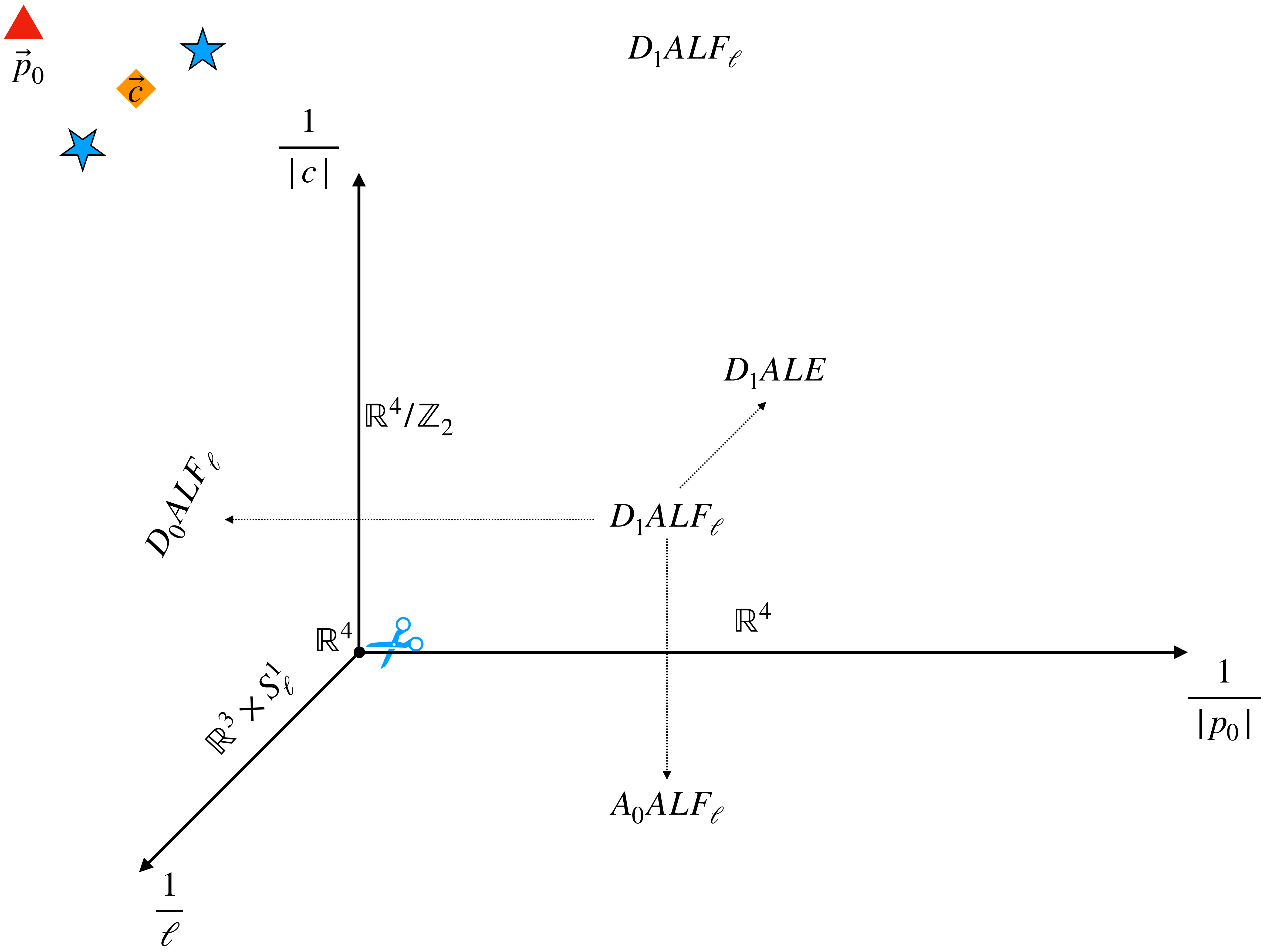


$\vec{p}_0$
$\vec{c}$
$\vec{p}_1$
D_2ALF
$\frac{1}{|c|}$
D_1ALF
D_1ALF
$(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$
$\mathbb{R}^3 \times S^1$
A_0ALF
$\frac{1}{|p_0|}$
A_0ALF
$A_0ALF_{2\ell}/\mathbb{Z}_2$
A_1ALF
$\frac{1}{|p_1|}$

Relation to ALE



Moral: A_{k-1}- and D_k-ALE spaces are moduli spaces of monopoles of holomorphic charge 1 or centered monopoles of holomorphic charge 2 with k simple Dirac singularities.



$\vec{p}_0$
$\vec{c}$
D_1ALF_ℓ
$\frac{1}{|c|}$
D_1ALE
$\mathbb{R}^4/\mathbb{Z}_2$
D_0ALF_ℓ
D_1ALF_ℓ
$\mathbb{R}^4$
$\mathbb{R}^4$
$\mathbb{R}^3 \times S^1_\ell$
$\frac{1}{|p_0|}$
A_0ALF_ℓ
$\frac{1}{\ell}$

Two Monopoles with simple Dirac Singularities

D_k **ALE** 2 centered U(2) monopoles with non-max symm. breaking

on $\mathbb{R}^3$

$\lambda \to 0$ ↑

D_k **ALF**

2 centered U(2) monopoles with symm. breaking λ

on $\mathbb{R}^3$

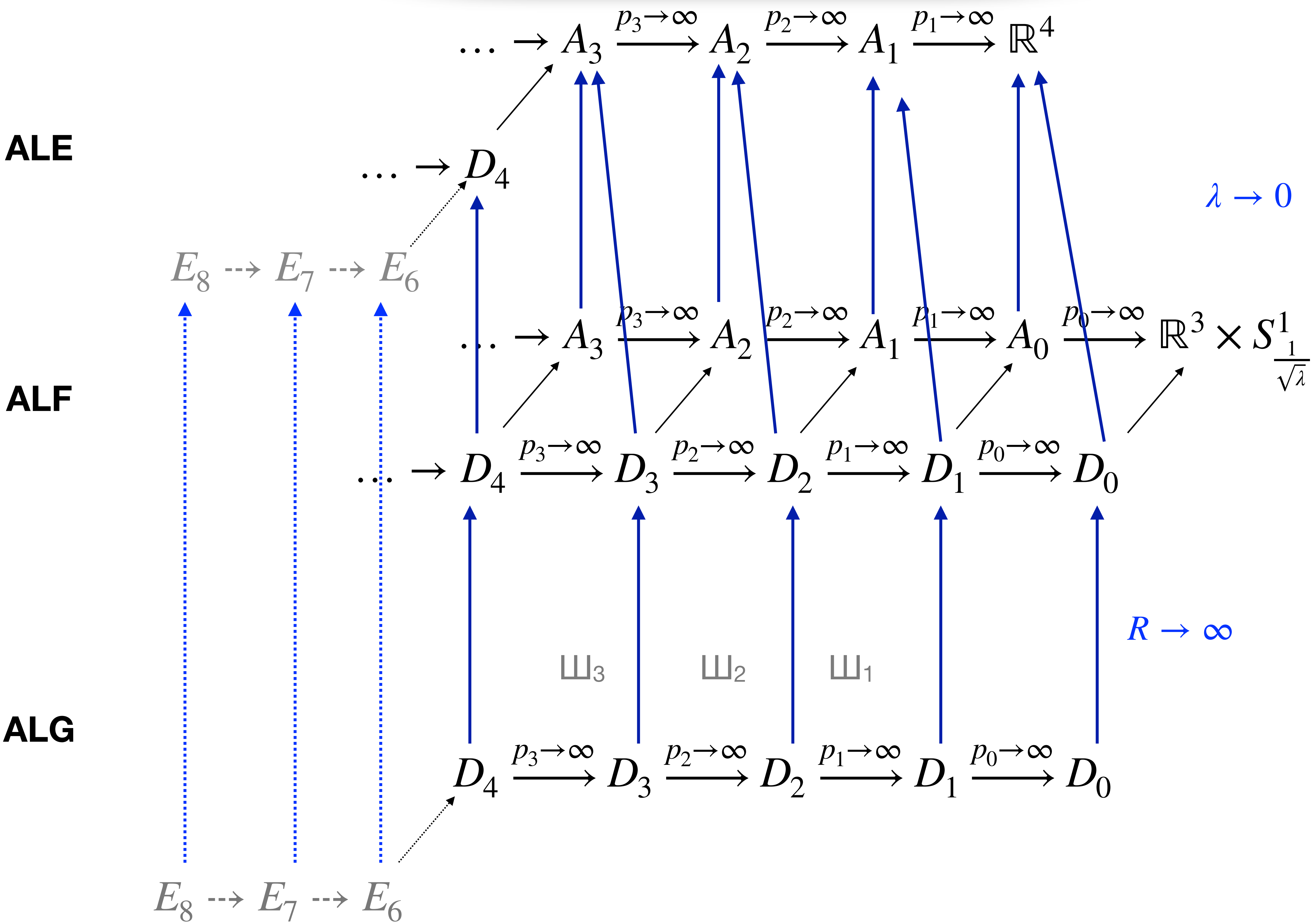
$R \to \infty$ ↑

D_0, D_1, D_2, D_3, D_4 **ALG**

2 centered U(2) monopoles with symm. breaking

on $\mathbb{R}^2 \times S^1_R$

Including ALG



Spectral View of Periodic Monopoles

Periodic Monopole = Monopole on

$$\underset{x,y}{\mathbb{R}^2} \times \underset{\varphi}{S^1} = \mathbb{C} \times S^1$$

$$z = x + iy$$

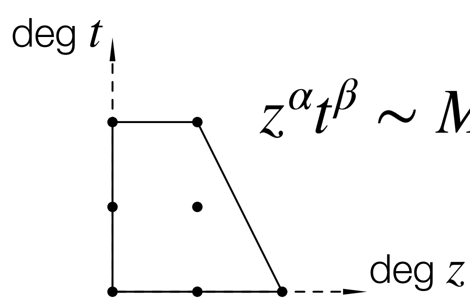
Monowall = Monopole on

$$\underset{x}{\mathbb{R}} \times \underset{\theta}{S^1} \times \underset{\varphi}{S^1} = \mathbb{C}^* \times S^1$$

$$s = \exp \frac{x + i\theta}{2\pi R_\theta}$$

Eigenvalues of monodromy of $\nabla^A_\varphi + \Phi$ around the S^1 factor form an algebraic curve

$$\left\{ F(z,t) = 0 \mid z \in \mathbb{C}, t \in \mathbb{C}^* \right\}$$



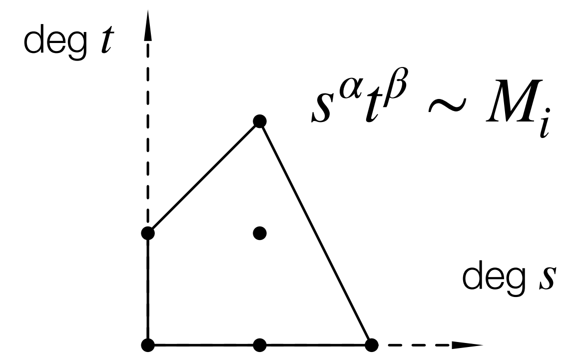
$$z^\alpha t^\beta \sim M_i$$

$$\Phi \sim \frac{1}{2\pi R_\varphi}\left(\frac{\alpha}{\beta} \ln|z| + \frac{1}{\beta} \ln M_i\right)$$

Subleading $\frac{1}{z}$ term deformation is NOT L^2!

Thus:
perim. & depth 1 coeffs = parameters
other coefficients = moduli

$$\left\{ P(s,t) = 0 \mid s \in \mathbb{C}^*, t \in \mathbb{C}^* \right\}$$



$$s^\alpha t^\beta \sim M_i$$

$$e^{2\pi R_\varphi (\Phi + iA_\varphi)} \sim t \sim M_i^{\frac{1}{\alpha}} s^{\frac{\beta}{\alpha}}$$

$$\Phi \sim \frac{1}{2\pi R_\varphi}\left(\frac{\alpha}{\beta} \underbrace{\frac{x}{2\pi R_\theta}}_{\ln|s|} + \frac{1}{\beta} \ln M_i\right)$$

All other deformations are L^2 !

Thus: perimeter coefficients = parameters
interior coefficients = moduli

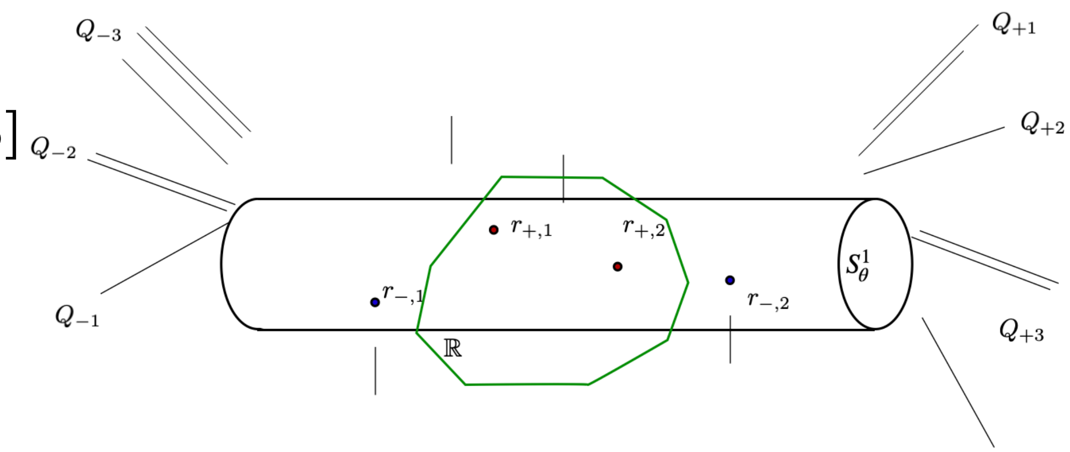
Monowalls

Ch-Ward '12
Ch '14
Ch-Cross '19

Monopole charges + singularities —> Newton polygon N

Number of moduli = 4 x Internal integer points of N

Number of parameters = 3 x [(Perimeter integer points of N) -3]

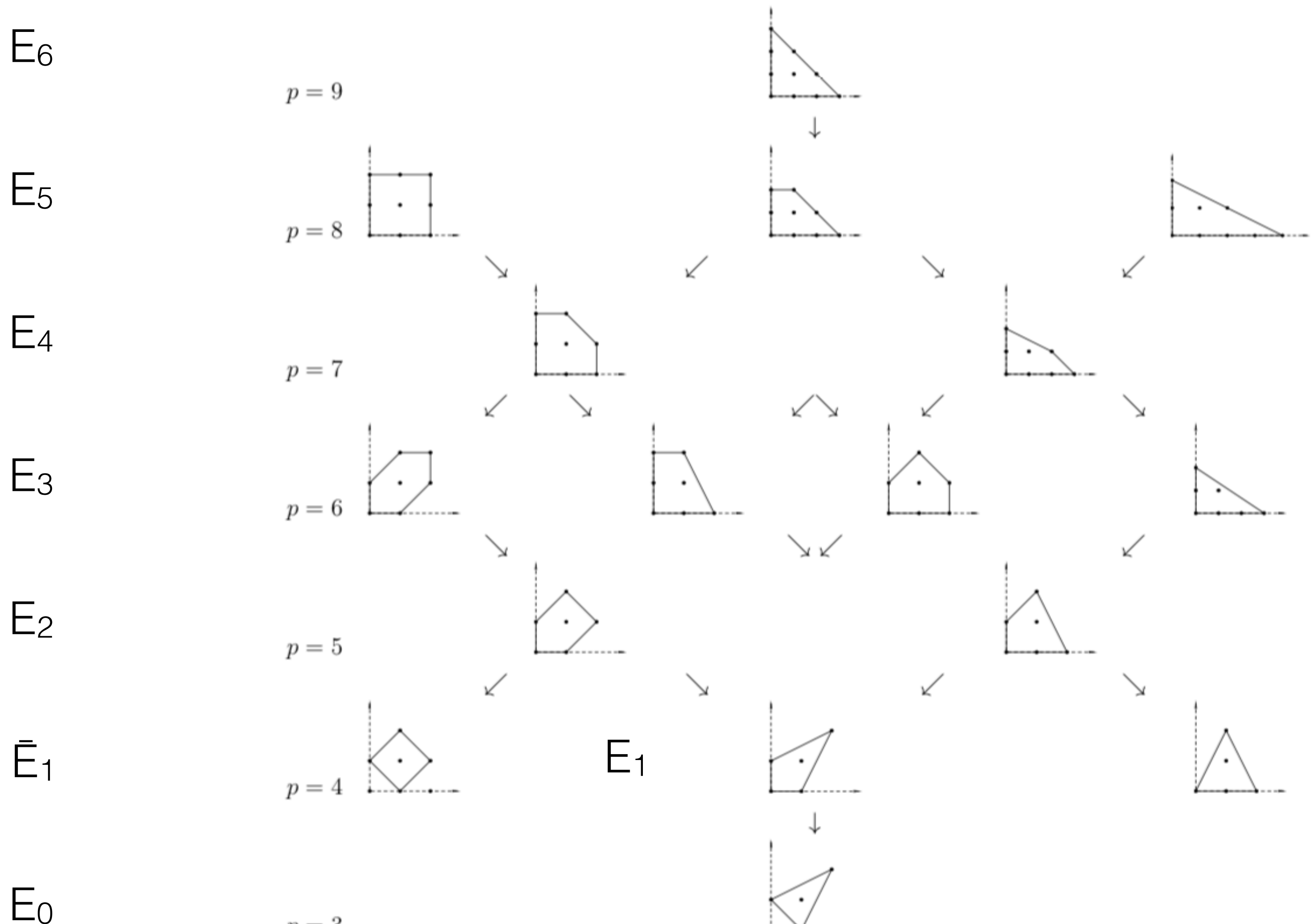


SL(2,$\mathbb{Z}$) moduli space isometry generated by

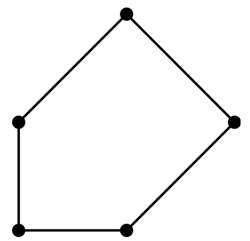
S = Nahm transform and

T= Adding constant magnetic field $(A, \Phi) \mapsto (A - \theta d\varphi, \Phi + x1)$.

All integer Newton polygons with a single internal point up to SL(2,$\mathbb{Z}$):



An example: E_2 ALH —> D_1 ALG



- Monowall with one simple Dirac singularity & charges (1,0) and (1,-1).

Spectral curve in $\mathbb{C}^* \times \mathbb{C}^*$:

$$st^2 - e^M(s^2 - us + 1)t + s - p = 0$$

- Limit opening one periodic direction: $R_\theta \to \infty$

$s = e^{\frac{z}{R_\theta}}$

$p = e^{\frac{q}{R_\theta}}$

let $u = 2 + \frac{v}{R_\theta^2}$ rescaled modulus

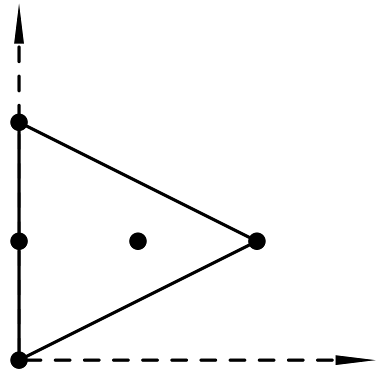
and $e^M = R_\theta^{\frac{3}{2}} e^{\tilde{M}}$ rescaled parameter

and $t = \frac{\tilde{t}}{\sqrt{R_\theta}}$ constant shift in Higgs field

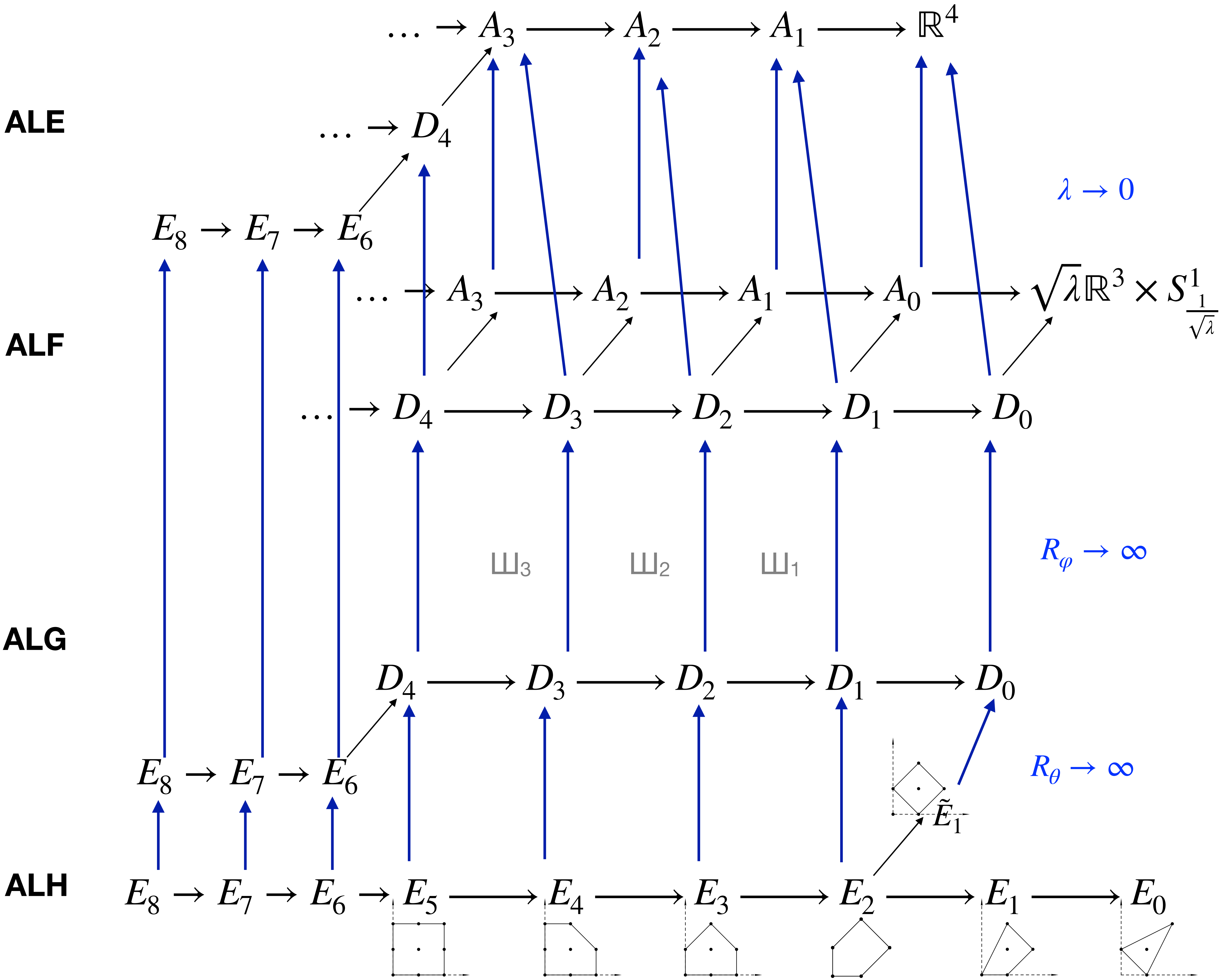
- Limiting spectral curve in $\mathbb{C} \times \mathbb{C}^*$

$$\tilde{t}^2 - e^{\tilde{M}}(z^2 - v)\tilde{t} + z - q = 0$$

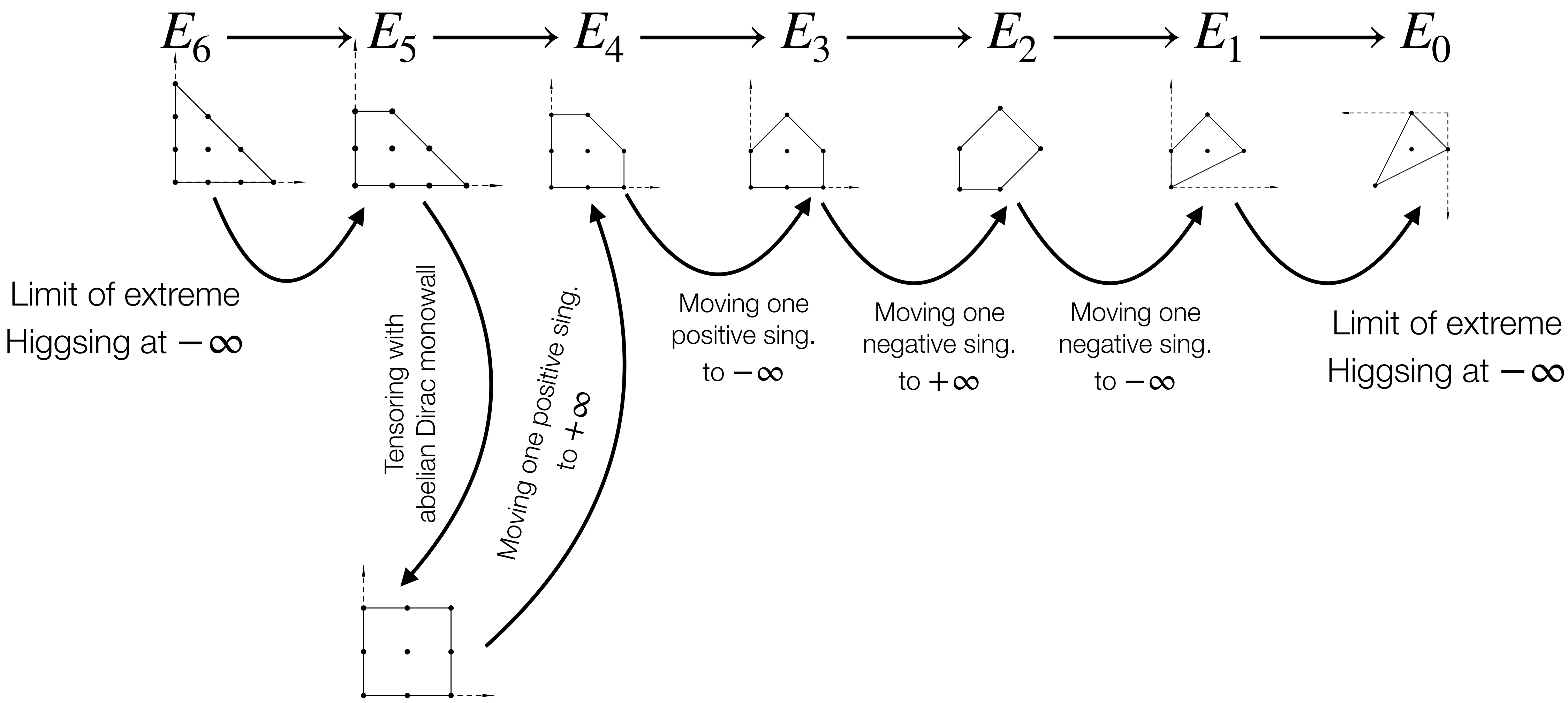
Periodic monopole of nonabelian charge 2 with one singularity.



In general: E_{k+1} ALH —> D_k ALG


… → A3 → A2 → A1 → R4
ALE
… → D4
λ → 0
E8 → E7 → E6
… → A3 → A2 → A1 → A0 → √λ R3 × S1 1/√λ
ALF
… → D4 → D3 → D2 → D1 → D0
Ш3
Ш2
Ш1
Rφ → ∞
ALG
D4 → D3 → D2 → D1 → D0
E8 → E7 → E6
Ẽ1
Rθ → ∞
ALH
E8 → E7 → E6 → E5 → E4 → E3 → E2 → E1 → E0

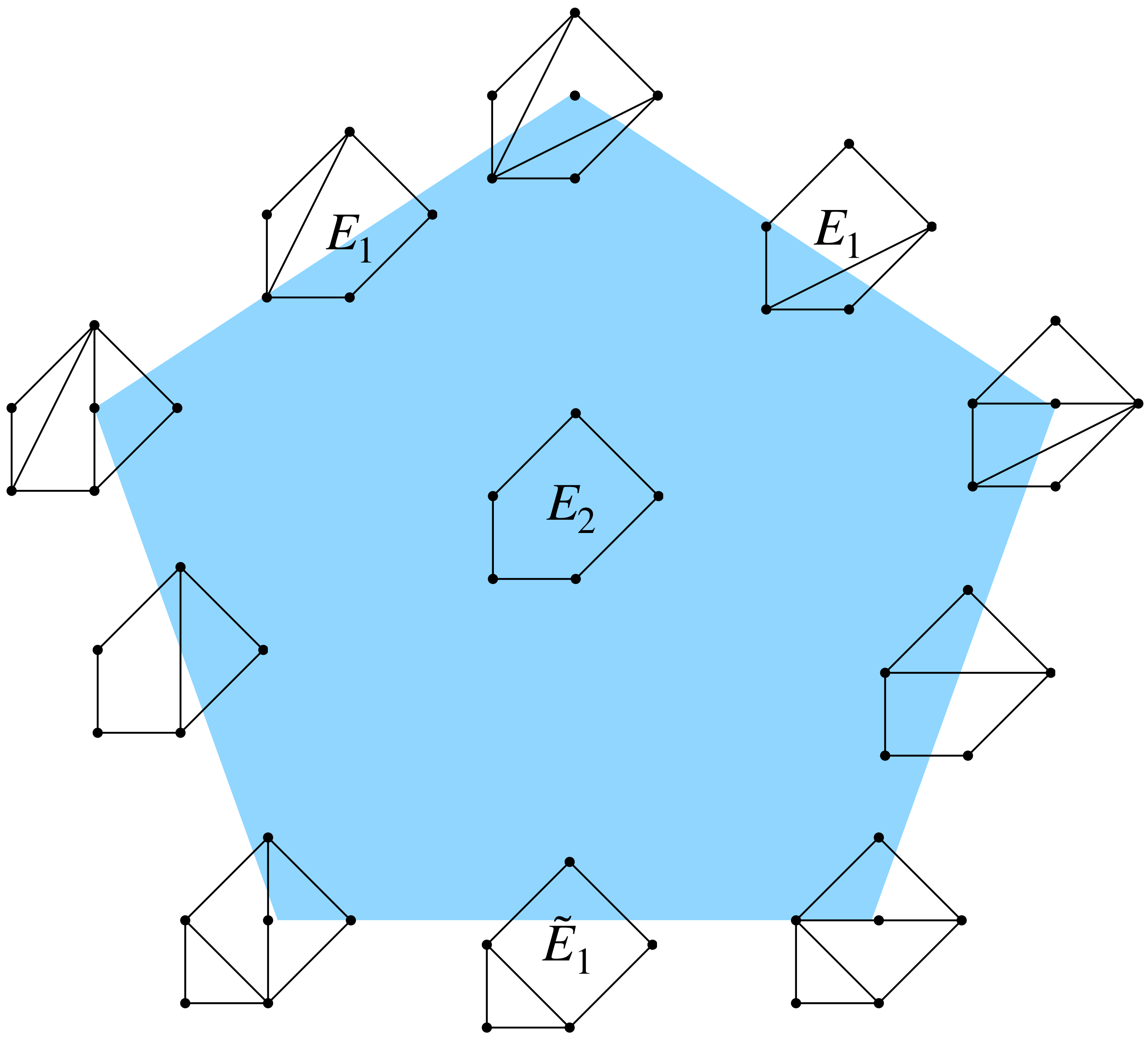
Horizontal ALH Relations



Space of all ALH metrics

The parameter space of ALH metrics is fibered over the “universal ALH associahedron”.

For example:



E_7 and E_8 ALH

These two cases were missing from the list of Monowalls with simple Dirac singularities.

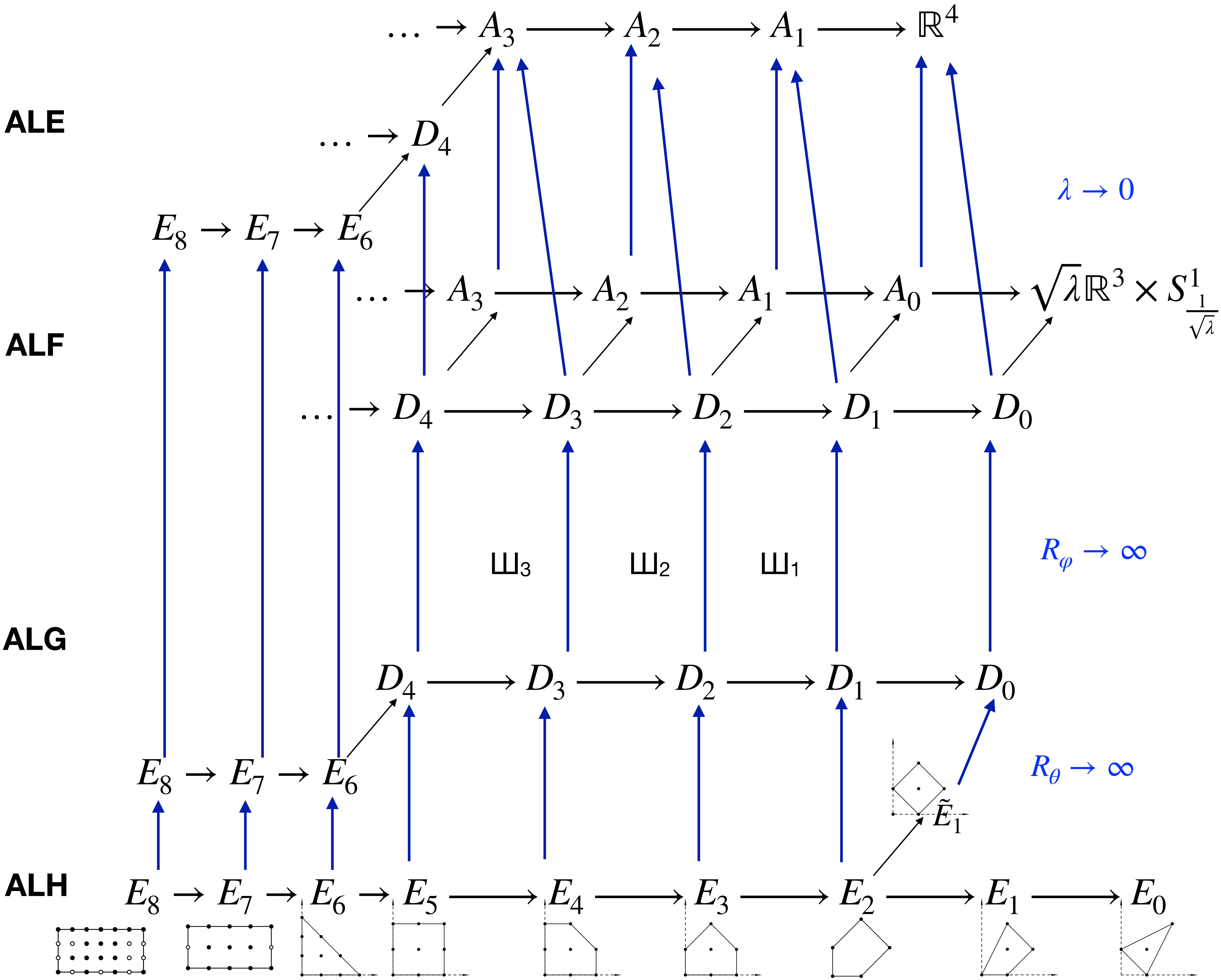
Thomas Harris identified these as moduli spaces of

a) monowalls with more complicated Dirac singularities and

b) monowalls with non-maximal symmetry breaking at infinity!

This breakthrough allows to describe ALG and ALE limits.

Moral: E-type ALG and even ALE spaces are
moduli spaces of singular monopoles.


… → A3 → A2 → A1 → ℝ4
ALE
… → D4
λ → 0
E8 → E7 → E6
… → A3 → A2 → A1 → A0 → √λ ℝ3 × S1 1/√λ
ALF
… → D4 → D3 → D2 → D1 → D0
Rφ → ∞
Ш3
Ш2
Ш1
ALG
D4 → D3 → D2 → D1 → D0
E8 → E7 → E6
Rθ → ∞
Ẽ1
ALH
E8 → E7 → E6 → E5 → E4 → E3 → E2 → E1 → E0

Ш-type ALG

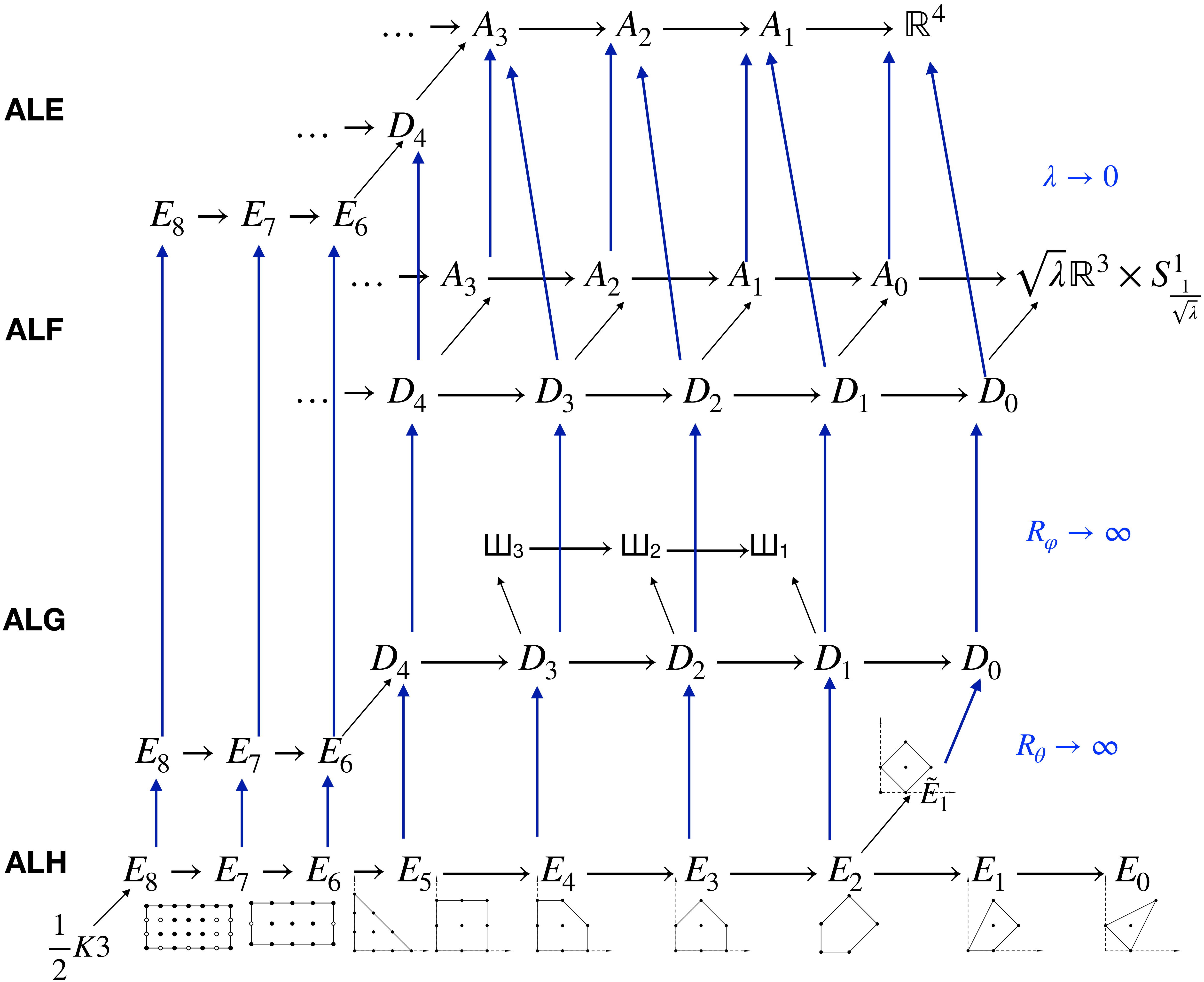
Now it is time for Ш-type ALG spaces.

Each is a limit of a D-type ALG space!

D_1 ALG —> $Ш_1$ ALG

D_2 ALG —> $Ш_2$ ALG

D_3 ALG —> $Ш_3$ ALG


ALE
ALF
ALG
ALH
… → A3 → A2 → A1 → ℝ4
… → D4
E8 → E7 → E6
… → A3 → A2 → A1 → A0 → √λℝ3 × S1 1/√λ
… → D4 → D3 → D2 → D1 → D0
λ → 0
Rφ → ∞
Ш3 → Ш2 → Ш1
D4 → D3 → D2 → D1 → D0
E8 → E7 → E6
Rθ → ∞
Ẽ1
E8 → E7 → E6 → E5 → E4 → E3 → E2 → E1 → E0
1/2 K3

Future problems & generalizations

- Once every tesseron is a monopole moduli space,
  every hyperkähler manifold is likely to be a monopole moduli space as well.

- For each tesseron find a construction of Yang-Mills instantons on it (generalizing the ADHM-Nahm transform).

- From the monopole limiting procedure deduce relations between different Quives, Nahm, Hitchin, etc systems.